# Adverse selection without single crossing: monotone solutions

Christoph Schottmüller

*Department of Economics, University of Copenhagen, Øster Farimagsgade 5, building 26, 1353 Copenhagen K, Denmark*

**Abstract**

The single-crossing assumption simplifies the analysis of screening models as local incentive compatibility becomes sufficient for global incentive compatibility. If single crossing is violated, global incentive compatibility constraints have to be taken into account. This paper studies monotone solutions in a screening model that allows a one-time violation of single crossing.

The results show that local and non-local incentive constraints distort the solution in opposite directions. Therefore, the optimal decision might involve distortions above as well as below the first-best decision. Furthermore, the well-known "no distortion at the top" property does not necessarily hold. The results show that the decision can even be distorted above first best for all types. Sufficient conditions for existence, (strict) monotonicity and continuity of the solution are presented. A new necessary condition satisfied by such solutions is found. An algorithm based on this condition can calculate continuous and strictly monotone solutions.

*JEL classification:* C61, D82, D86

*Keywords:* Spence-Mirrlees condition, global incentive compatibility, screening

## 1. Introduction

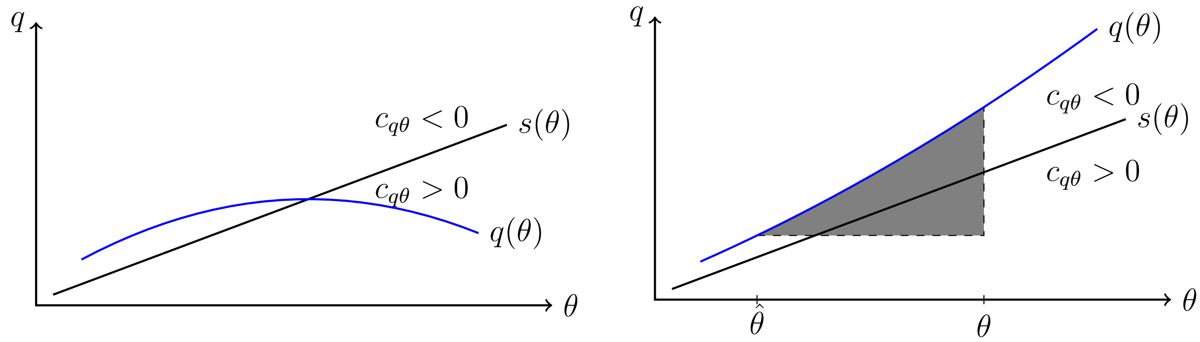
Screening models are among the most commonly used tools in microeconomics. In these models, a principal offers a menu of contracts from which an agent with private information about his "type" chooses his preferred option. Depending on the application, the type represents the production technology of a firm in models of regulation [1, 2], the productivity of a worker in an employment relationship [3] or the willingness to pay for a product in models of monopoly pricing [4].

In all these applications, the authors assume single crossing (SC):[1] types can be ordered according to their marginal rate of substitution between money and the decision, e.g. the quantity purchased in a monopoly pricing problem or the quantity produced in a regulation setup. With the commonly used quasilinear preferences, SC is equivalent to a type ordering according to marginal utilities/costs; e.g. a higher type has a higher marginal utility at every consumption level. The private information of an agent is then his eagerness to consume more.

While SC is a technically convenient assumption, we can think of many unobserved differences other than eagerness/efficiency. Some people are very eager at first but quickly saturated while others have more steady preferences. Compare, for example, single-person households with families. At low quantities, single persons might have a higher marginal willingness to pay for standard groceries because of a higher income per household member. At high quantities, however, families will have the higher marginal willingness to pay as single persons are already saturated. This violates SC: household types cannot be ordered according to their marginal willingness to pay.

*Email address:* christoph.schottmueller@econ.ku.dk (Christoph Schottmüller)

[1]Other names for this assumption include "Spence-Mirrlees" or "constant sign" condition.




(a) inverse U-shape solution

(b) monotone solution

Figure 1: possible solution shapes

As a second example, take firms with private information about their production technology. A capital-intensive, fully-automated production facility will – at normal output levels – have lower marginal costs than a labor-intensive technology. But as soon as quantity approaches the capacity level of the capital-intensive facility, the labor-intensive production technology could have lower marginal costs.

This paper analyzes a screening model in which SC is violated. Agents have quasilinear preferences and a one-dimensional type. SC would require a constant sign of the cross derivative of the agent's cost function with respect to type and decision. By contrast, my setting allows for a one-time violation of SC: the type versus decision plane is separated by a strictly increasing function $s$ into two regions. The cross derivative is negative above $s$ and positive below $s$ (see figure 1).

The revelation principle implies that implementable menus of contracts can be viewed as mechanisms in which the agent is induced to truthfully reveal his type. Put differently, the menu must be incentive compatible. With SC, the local first- and second-order conditions of the agent's maximization problem (maximizing utility over his type announcement) are necessary and sufficient for incentive compatibility. Without SC, these first- and second-order conditions are necessary but not sufficient for incentive compatibility.

Incentive compatibility requires the optimal decision function $q(\theta)$ to be increasing (decreasing) whenever $c_{q\theta}(q(\theta),\theta) < (>) 0$ and $q$ is continuous at $\theta$. A continuous optimal decision function (decision as function of type) will therefore be either inversely U-shaped or monotone. Araujo and Moreira [5] analyze solutions in which the decision function is inversely U-shaped, see figure 1a. In this case, some contracts are chosen by two types and consequently the incentive compatibility constraint is satisfied with equality between any two such types. I analyze monotone solutions, that is, situations in which higher types are assigned a higher decision under the optimal mechanism, see figure 1b. In monotone solutions, a type can be indifferent between two distinct contracts. I provide a new necessary condition satisfied by strictly monotone and continuous solutions and introduce an algorithm to calculate such solutions. Sufficient conditions for the existence of a (strictly) monotone and continuous solution are derived.

If SC holds, then there is no distortion at the top and the distortion for all types goes in the same direction, e.g. all types consume/produce a quantity weakly below their first-best quantity. When SC is violated, neither result has to hold.

The reason is that binding non-local incentive constraints will counteract the normal distortion stemming from rent extraction motives. A rough intuition for this result is the following: suppose the principal took only the first and second-order conditions of the agent's maximization problem into account when designing the contract menu. The resulting decision function will be distorted below first best for almost all types because these distortions allow the principal to extract rents from the agent. If SC is violated, this resulting menu might, however, not be implementable; a type $\theta$ might benefit from choosing the contract of another type, say $\hat{\theta} < \theta$, instead of choosing the contract intended for him. To satisfy the incentive constraint of type $\theta$, the principal must leave a higher rent to type $\theta$ (when choosing the contract intended for him). Reducing the normal distortion – or even distorting the decision in the opposite direction – will result in less rent extraction, leave a higher rent to type $\theta$ and, therefore, relax the incentive constraint.

The effect of non-local incentive constraints can be so strong that, in contrast to the previous literature, *all* types' decisions are distorted. This result could, for example, rationalize flat rate tariffs where a zero marginal price incentivizes overconsumption.

The following subsection gives an example of preferences that violate single crossing. The related literature is reviewed in subsection 1.2 and the formal model is introduced in section 2. Section 3 shows that the distortion stemming from non-local incentive constraints counteracts the usual distortion stemming from local incentive constraints. Furthermore, a first-order condition characterizing monotone solutions is derived. Section 4 focuses on strictly monotone and continuous solutions and provides a new necessary condition satisfied by such solutions. This condition makes it possible to formulate an algorithm to determine such solutions. In order to apply the algorithm, one has to know that a continuous and strictly monotone solution exists. Therefore, section 5 gives sufficient conditions for these solution properties. A numerical example in which the decision of all types is distorted is presented in section 6. Section 7 discusses how the results depend on the assumptions as well as possible generalizations. Most proofs are relegated to the appendix. A webappendix includes more numerical examples and those proofs that are straightforward extensions of other proofs in the paper or in the literature.

### *1.1. Example setting where single crossing is violated*

This section introduces a numerical example which is used for illustration later on. The webappendix contains more examples.

Consider a setting of monopoly regulation. The monopolist produces a good with two input factors which he uses in fixed proportions. The costs of the first input factor are proportional to output whereas the costs of the second input factor are convexly increasing in output. One interpretation could be that the first input is unskilled labor which can be hired in any quantity on a competitive labor market while the second input is skilled labor which is increasingly difficult to hire. The monopolist's private information is whether his production technology is more intense in the first or second factor. A cost function that captures these ides is

$$c(q,\theta) = \theta q + \frac{q^2}{\theta} + \gamma(\theta)$$

where $\gamma(\theta)$ are fixed costs, $q$ is quantity and $\theta$ is the type. The cross partial derivative $c_{q\theta}(q,\theta) = 1 - 2q/\theta^2$ can change sign and therefore SC is violated: marginal costs can be in- or decreasing in type (depending on the value of $q$). For low quantities, the linear part of the cost function dominates marginal costs and therefore high types have higher marginal costs. For high quantities, the convex part of the cost function is more relevant and therefore high types have lower marginal costs.

The cost function in this example is a simplified version of the flexible fixed cost quadratic cost function suggested by [6]. Such a cost function is estimated in [7] for savings and loans associations. Interestingly, they allow for two unobservable types ("mixtures" in their language) of production technology in their estimation. It follows from their estimates in table 5 that mixture 1 has lower marginal costs at low output levels but higher marginal costs at high output levels. Hence, estimated cost functions in the savings and loan sector violate SC.

The example above illustrates a more general point: to avoid the difficulties of multidimensional screening, some papers collapse a multidimensional type into a one-dimensional type ; e.g. we could have used the cost function $\theta_1 q + \theta_2 q^2$ instead of the one above and assumed that $\theta_1$ and $\theta_2$ are negatively correlated. When there is a trade-off between being marginally efficient in either one or the other dimension, this collapsing will lead to a setting where SC is violated. Non-local incentive constraints will also play a role in the corresponding multidimensional model. The multidimensional screening literature mainly sidesteps this complication, for example, by using linear utility/cost functions as described below.

### *1.2. Literature*

Closest to this paper is Araujo and Moreira [5]. Their paper characterizes (inversely) U-shaped solutions in a setup with a one-time violation of SC. My paper complements their work by characterizing monotone solutions in the same model. Whether the optimal contract menu is monotone or (inversely) U-shaped depends on the utility functions of the agents. Roughly speaking, the shape of the solution is determined by the shape of the first-best decision; see section 5.1 for details.

The main technical difference is that incentive constraints in the current paper can bind between types choosing different contracts from the menu. Qualitatively, the solution in [5] features either a discontinuity or a bunching interval. Furthermore, there is a no distortion at the top result, so the type with the highest first-best decision[2] will be assigned this very decision under the optimal mechanism. This paper shows that monotone solutions can be strictly monotone and continuous and therefore bunching and discontinuities are not a necessary implication of a violation of SC. Furthermore, I show that distortion at the top and even distortion of all types' decision is possible in monotone solutions. The (inverse) U-shape solution and its critical condition (see section 3.1 and the webappendix) has been applied in an insurance model [8], in non-linear monopoly pricing [9] and signaling games [10].

Several authors have analyzed perfect competition insurance models with discrete types where SC is violated. In their papers, private information has two dimensions and can take either a high or a low value in each dimension. This means there are $2 \times 2$ types. In Smart [11], the two dimensions are risk and risk aversion while in Wambach [12] they are wealth and risk. Netzer and Scheuer [13] model an additional labor supply decision and the two dimensions are productivity and risk. All three papers share a pooling result: two of the four types can be pooled if SC is violated. In contrast to my paper, these discrete type models have no distortion at the top and decisions of all other types are distorted downward.

Violations of SC are also related to the literature on multidimensional screening, see Armstrong [14] and Rochet and Choné [15] for seminal contributions, Rochet [16] for a recent related paper and Rochet and Stole [17] for a survey. As pointed out in the survey, "the problems arise not because of multiple dimensionality itself, but because of a commonly associated lack of exogenous type-ordering in multiple-dimensional environments." A violation of SC also conveys a lack of type-ordering. To make the relationship clear, think of a multidimensional, discrete type model. Clearly, one can reassign types to a one-dimensional parameter but with those reassigned types SC will normally be violated.

Generally speaking, multidimensionality leads to two new challenges concerning incentive compatibility. First, it is ex ante unclear which local incentive constraints are binding. Second, non-local incentive constraints can be binding. The literature has mainly focused on the first challenge[3] and provided sufficient conditions ensuring that non-local incentive constraints are not binding (see for example Carroll [20]). My paper analyzes a model where non-local incentive constraints are binding. Although the model is one-dimensional, this can be viewed as a natural starting point for solving more general setups.

The paper also relates to work relaxing the basic assumptions of the textbook screening model [21, 22]. Jullien [23] allows for type dependent participation constraints while Hellwig [24] analyzes the case of irregular type distributions, i.e. distributions with mass points and zero densities. Hellwig [24] shows that these irregularities lead to discontinuities as well as bunching in the optimal decision schedule. Contrary to my paper, there is no distortion above first best and no distortion at the top. In Jullien [23], distortion can be above as well as below first best. The reason is that incentive constraints can bind upward as well as downward. If a participation constraint binds in the interior, it is relaxed by increasing the decision of lower types. Consequently, the slope of the rent function increases at these lower types which leads to higher rents for the interior type. I show that binding incentive constraints can also lead to distortion above first best although incentive constraints are only downward binding. The intuition is that if a type $\theta$ wants to misrepresent as a lower type $\hat{\theta}$, one can relax this incentive constraint by increasing the decisions of types between $\hat{\theta}$ and $\theta$: doing so will increase the slope of the rent function and lead to higher rents for $\theta$ at his own contract making misrepresentation less attractive. Furthermore, I show that all types can have distorted decisions which is – to the best of my knowledge – a new result in the screening literature.

Another motivation for studying models where SC is violated is that violations of SC emerge naturally

[2] In Araujo and Moreira [5], the function $s(\theta)$ is downward sloping and the analyzed solution is actually U-shaped (not *inversely* U-shaped as it would be when applied to my setting). In their setting, the undistorted top type is then the type with the *lowest* decision.

[3] See Moore [18] and Matthews and Moore [19] for early exceptions in the discrete type case.

in some strands of the screening literature, for example in work on common agency or on advantageous selection. Authors have – so far – restricted themselves to analyze models with specific functional forms, typically quadratic utility functions and uniformly distributed types, for which they could show that the agent's first-order condition is sufficient for incentive compatibility, [25, 26, 27, 28]. In the signaling literature, SC fails in models where agents' signaling is motivated by social image concerns, [29, 30, 31]. Consequently, violations of SC are also likely to matter in non-linear pricing problems with status goods.

## 2. Model

A principal and an agent contract on a decision $q \in \mathbb{R}_+$ and a monetary transfer $t \in \mathbb{R}$. The agent's utility is $\pi = t - c(q,\theta)$ where $\theta \in \Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ is the type of the agent which is his private information. The function $c(q,\theta)$ is assumed to be three times continuously differentiable with $c_\theta < 0$. The outside option is assumed to be zero for all types. The participation constraint can therefore only bind for the lowest type. Hence, any deviation from the standard solution will not be due to participation constraints binding in the interior, see Jullien [23] for this, but due to the violation of single crossing.

The principal's utility is $u(q,\theta) - t$ and is two times continuously differentiable. The principal has a prior $F(\theta)$ with continuous density $f(\theta) > 0$ for all $\theta \in [\underline{\theta}, \bar{\theta}]$. To simplify the exposition, I assume full participation, i.e. the surplus from trade is so high that it is not beneficial to exclude some types.[4]

The following assumptions are standard in the screening literature. They would ensure that the problem is well behaved if SC was satisfied. Section 7 discusses how the results of the paper extend when these assumptions are violated.

**Assumption 1.** *The principal's utility function satisfies $u_{q\theta} \geq 0$ and $u_{qq} \leq 0$. The agent's cost function satisfies $c_{qq} > 0$, $c_{qq\theta} < 0$ and $c_{q\theta\theta} > 0$. For all $\theta \in \Theta$, there exists an $s(\theta)$ defined by $c_{q\theta}(s(\theta),\theta) = 0$ which is strictly greater than 0. F satisfies the monotone hazard rate property: $f/(1-F)$ is increasing.*[5]

By making assumptions on the third derivatives of the cost function, I allow for a one-time violation of SC; for a given type, $c_{q\theta}$ can change sign once when varying the decision $q$ (see figure 1). Hence, the type decision plane is divided into two regions; one where $c_{q\theta} < 0$ and one where $c_{q\theta} > 0$. These two regions are separated by a strictly increasing function $s$ defined by $c_{q\theta}(s(\theta),\theta) = 0$. It is assumed that $s > 0$, so that the violation of SC occurs for all types.

By the revelation principle, any contract menu can be mimicked by a direct revelation mechanism in which the agent truthfully reports his type [32]. A direct revelation mechanism consists of a pair of functions $(q,t) : \Theta \to \mathbb{R}_+ \times \mathbb{R}$ to which the principal commits.[6] The interpretation is that the agent announces a type $\hat{\theta}$ and this type announcement results in the allocation $(q(\hat{\theta}), t(\hat{\theta}))$. The agent truthfully reports his type if the *incentive compatibility constraint*

$$t(\theta) - c(q(\theta),\theta) \geq t(\hat{\theta}) - c(q(\hat{\theta}),\theta) \quad \text{for all } \theta, \hat{\theta} \in \Theta \tag{1}$$

is satisfied. In this case, the decision function $q : \Theta \to \mathbb{R}_+$ is said to be *implementable* through the transfer function $t : \Theta \to \mathbb{R}$. A contract $(q,t)$ is *feasible* if it satisfies the agent's individual rationality constraint

$$t(\theta) - c(q(\theta),\theta) \geq 0 \quad \text{for all } \theta \in \Theta. \tag{IR}$$

A type $\theta$ agent will maximize $t(\hat{\theta}) - c(q(\hat{\theta}),\theta)$ over his type announcement $\hat{\theta}$. This leads to the necessary first and second-order conditions of the incentive compatibility constraint.

[4]This assumption is less restrictive than it might seem. By $c_\theta < 0$, only types at the low end could be excluded. If exclusion is optimal, the characterization in this paper applies to the set of not excluded types. Using the methods of this paper, one can calculate the solution for any given cutoff type and then maximize the principal's payoff over the cutoff type.

[5]Throughout the paper, a function $f$ is called *increasing* if $x > y$ implies $f(x) \geq f(y)$.

[6]I will concentrate on deterministic mechanisms. The webappendix gives a sufficient condition under which deterministic mechanisms are optimal.

**Lemma 1.** *Let $q$ be a decision function implementable through transfers $t$. Then, the following conditions hold:*

- *Envelope Theorem/First order condition: the agent's rent function is given by*

$$\pi(\theta) \equiv t(\theta) - c(q(\theta), \theta) = \pi(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} c_\theta(q(x), x)\, dx \quad \text{for all } \theta \in \Theta. \tag{2}$$

- *Monotonicity/Second order condition: let $q$ be right continuous with left-hand side limit at $\theta$. If $q(\theta) > (<) s(\theta)$, then $q^-(\theta) \leq (\geq) q^+(\theta)$ and $q$ is increasing (decreasing) on $(\theta, \theta + \varepsilon)$ for $\varepsilon > 0$ small enough.*[7]

- *$q$ is bounded from above.*

One implication of the envelope condition that will be useful later on is that $\pi_\theta(\theta) = -c_\theta(q(\theta), \theta)$ for almost all types. The monotonicity condition in lemma 1 implies that the optimal decision function cannot have arbitrary shapes. Araujo and Moreira [5] analyze inverse U-shape decisions as depicted in figure 1a. In such a solution, one decision can be assigned to two types ("discrete pooling") and therefore these two types have the same contract. Clearly, (1) holds with equality between such two types. It turns out that discretely pooled types are the only types for which (1) holds with equality. I will analyze monotone solutions in this paper as depicted in figure 1b.[8]

The violation of SC plays a role in monotone solutions although $c_{q\theta}(q(\theta), \theta) < 0$ for all types. To see this, rewrite (1) using (2) as

$$c(q(\hat\theta), \theta) - c(q(\hat\theta), \hat\theta) - \int_{\hat\theta}^{\theta} c_\theta(q(x), x)\, dx \geq 0.$$

This inequality can again be rewritten as

$$\Phi(\theta, \hat\theta) \equiv \int_{\hat\theta}^{\theta} \left( c_\theta(q(\hat\theta), x) - c_\theta(q(x), x) \right) dx = - \int_{\hat\theta}^{\theta} \int_{q(\hat\theta)}^{q(x)} c_{q\theta}(y, x)\, dy\, dx \geq 0. \tag{NLIC}$$

Note that every $q$ satisfying (NLIC) for all types $\theta$, $\hat\theta \in \Theta$ is implementable through transfers given (up to a constant $\pi(\underline{\theta})$) by (2). It follows from (NLIC) that incentive compatibility between two types can be represented as an integral over the shaded area in figure 1b: If the integral of $c_{q\theta}$ over this shaded area is negative, then incentive compatibility is satisfied for $\theta$ and $\hat\theta$. Hence, the region where $c_{q\theta} > 0$ is relevant for incentive compatibility although the solution does not pass through it.

(NLIC) also shows that the envelope theorem and monotonicity are sufficient for incentive compatibility if SC is satisfied. SC means that the integrand in (NLIC) has the same sign for any argument. Monotonicity of $q$ implies then the inequality (NLIC).

The principal's second-best problem is to maximize

$$\int_{\underline{\theta}}^{\bar\theta} \left( u(q(\theta), \theta) - t(\theta) \right) f(\theta)\, d\theta$$

over $q(\cdot)$ and $t(\cdot)$ subject to incentive compatibility and individual rationality. Using (2) to eliminate $t$ from the principal's objective and integrating by parts transforms the principal's objective into the usual virtual valuation and leads to the following second best problem:

$$\max_{q(\cdot), \pi(\underline{\theta})} \int_{\underline{\theta}}^{\bar\theta} \left( u(q(\theta), \theta) - c(q(\theta), \theta) \right) f(\theta) + (1 - F(\theta)) c_\theta(q(\theta), \theta)\, d\theta - \pi(\underline{\theta}) \qquad \text{s.t. } (NLIC) \text{ and } (IR). \tag{SB}$$

[7] The superscript "–" ("+") indicates limits from below (above).

[8] I will focus on monotonically increasing solutions. It is easy to show that in solutions that are below $s(\theta)$ for all types (and therefore are decreasing) non-local incentive constraints are slack. Hence, standard methods are sufficient to solve such problems.

The space of decision functions is assumed to be the space of measurable functions. The individual rationality constraint can only bind for $\underline{\theta}$ because of the assumption $c_\theta < 0$. As leaving rents to the agent is costly to the principal, (IR) will bind for $\underline{\theta}$ and therefore $\pi(\underline{\theta}) = 0$ in the optimal contract menu.

I will refer to the constraint (NLIC) in problem (SB) as the *non-local incentive compatibility constraint.* If (NLIC) binds such that type $\theta$ is indifferent between his contract and the contract of $\hat{\theta}$, I will say that (NLIC) is *binding from $\theta$ to $\hat{\theta}$.* I will use the term *local incentive compatibility constraint* to refer to the envelope condition in lemma 1.

Before turning to the analysis of the solution, I define two benchmarks. The *first-best* decision denoted by $q^{fb}(\theta)$ is defined as $argmax_q u(q,\theta) - c(q,\theta)$ which is the decision maximizing joint surplus. Existence and uniqueness of $q^{fb}(\theta)$ follow from concavity of $u - c$ (assumption 1). As a second benchmark, the *relaxed solution* $q^r$ is defined as the solution of the following *relaxed program* which neglects the constraint (NLIC):

$$\max_{q(\cdot)} \int_{\underline{\theta}}^{\bar{\theta}} \left(u(q(\theta),\theta) - c(q(\theta),\theta)\right) f(\theta) + (1 - F(\theta)) c_\theta(q(\theta),\theta)\, d\theta. \tag{RP}$$

The inverse U-shape solutions in [5] occur when $q^{fb}$ is inversely U-shaped and intersects $s$. In order to focus on monotone solutions, I require the first-best decision to be above $s$. This will – together with assumption 1 – imply that $q^{fb}$ and $q^r$ are strictly monotone.

**Assumption 2.** *$s(\theta) < q^{fb}(\theta)$ for all $\theta \in \Theta$.*

**Lemma 2.** *The first-best decision is strictly increasing. The relaxed program is strictly concave. The relaxed decision is strictly increasing and lies strictly above $s$.*

Lemma 2 implies that the relaxed decision $q^r$ is implicitly defined by the pointwise first-order condition

$$\left(u_q(q(\theta),\theta) - c_q(q(\theta),\theta)\right) f(\theta) + (1 - F(\theta)) c_{q\theta}(q(\theta),\theta) = 0. \tag{3}$$

If SC was satisfied, the relaxed decision would be the solution of the principal's problem. Without SC, non-local incentive constraints can be violated under the relaxed decision. The following section will show how the solution qualitatively differs from $q^r$ as a result of binding non-local incentive constraints.

## 3. Monotone solutions

### *3.1. Necessary conditions*

This subsection presents necessary conditions that must be met whenever (NLIC) holds with equality. Since these conditions are only a slight generalization of those presented in theorem 1A and 2 of Araujo and Moreira [5], the presentation will be brief and more intuitive than formal.

Take an optimal decision function $q$ and assume for now that $q$ is continuous and strictly increasing. Let NLIC be satisfied with equality for types $\theta$ and $\hat{\theta}$; i.e. $\Phi(\theta,\hat{\theta}) = 0$. $\Phi$ has to be non-negative for all types by (NLIC). Therefore, $(\theta,\hat{\theta}) \in argmin_{(x,y)}\Phi(x,y)$ as $\Phi(\theta,\hat{\theta}) = 0$. If $\hat{\theta} \neq \bar{\theta}$ and $\theta \neq \underline{\theta}$, the following first-order conditions for a minimum have to hold:[9]

$$\int_{q(\hat{\theta})}^{q(\theta)} -c_{q\theta}(y,\theta)\, dy \leq 0 \qquad \text{with “=” if } \theta < \bar{\theta}, \tag{C1}$$

$$\int_{\hat{\theta}}^{\theta} c_{q\theta}(q(\hat{\theta}),x)\, dx \geq 0 \qquad \text{with “=” if } \hat{\theta} > \underline{\theta}. \tag{C2}$$

Graphically, $\Phi(\theta,\hat{\theta}) = 0$ meant that the integral of $c_{q\theta}$ over the shaded area in figure 1b is zero. (C1) and (C2) imply that the integral of $c_{q\theta}$ over the right and the lower boundary of the shaded area in figure 1b is

[9]It will be shown later that the possibilities $\hat{\theta} = \bar{\theta}$ or $\theta = \underline{\theta}$ will not play a role in monotone decisions.

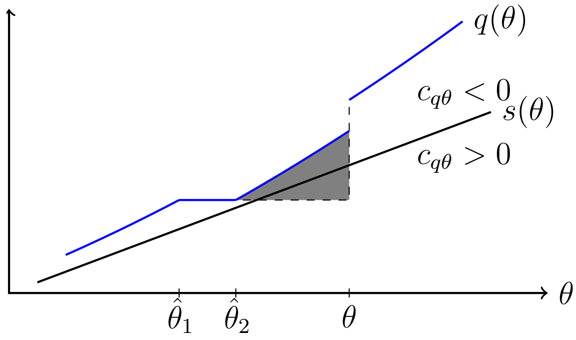



Figure 2: necessary conditions at discontinuity

zero. If, for example, the integral in (C1) was positive and $\Phi(\theta, \hat{\theta}) = 0$, then incentive compatibility would be violated for $\theta - \varepsilon$ and $\hat{\theta}$ as $\Phi(\theta - \varepsilon, \hat{\theta}) \approx \Phi(\theta, \hat{\theta}) - \varepsilon \int_{q(\hat{\theta})}^{q(\theta)} -c_{q\theta}(y, \theta)\, dy$.

The graphical interpretation also helps to quickly generalize these conditions to decision functions that are neither continuous nor strictly increasing. This situation is depicted in figure 2. Assume $\Phi(\theta, \hat{\theta}_i) = 0$ for $i = 1, 2$. To satisfy (NLIC) for types close to $\theta$, $\hat{\theta}_1$, and $\hat{\theta}_2$, the following conditions must hold:

- $\int_{q(\hat{\theta}_i)}^{q^-(\theta)} c_{q\theta}(y, \theta)\, dy \geq 0$ as otherwise $\Phi(\theta - \varepsilon, \hat{\theta}_i) < 0$,
- $\int_{q(\hat{\theta}_i)}^{q^+(\theta)} c_{q\theta}(y, \theta)\, dy \leq 0$ as otherwise $\Phi(\theta + \varepsilon, \hat{\theta}_i) < 0$,
- $\int_{\hat{\theta}_1}^{\theta} c_{q\theta}(q(\hat{\theta}), x)\, dx \leq 0$ as otherwise $\Phi(\theta, \hat{\theta}_1 - \varepsilon) < 0$,
- $\int_{\hat{\theta}_2}^{\theta} c_{q\theta}(q(\hat{\theta}), x)\, dx \geq 0$ as otherwise $\Phi(\theta, \hat{\theta}_2 + \varepsilon) < 0$.

Given (C1) and (C2), Araujo and Moreira [33] derive a third necessary condition for types at which (NLIC) holds with equality using variational calculus. While (C1) and (C2) are purely driven by incentive compatibility, this third condition will be derived from the principal's optimization. The idea is to have a variation of the optimal decision around $\theta$ and $\hat{\theta}$ such that the two necessary conditions (C1) and (C2) are still satisfied. The derivation is sketched in the webappendix. The following variational condition results for interior types $\theta$, $\hat{\theta}$ such that (NLIC) holds with equality and $q$ is strictly increasing in neighborhoods of $\theta$ and $\hat{\theta}$:

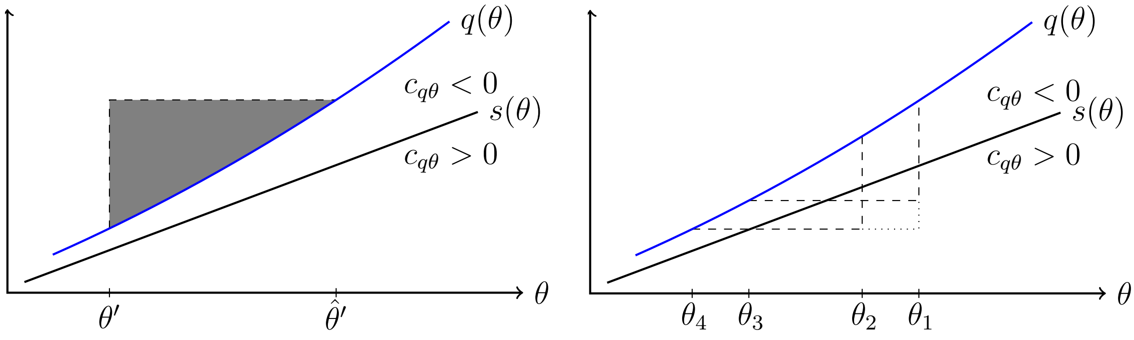
$$\frac{[u_q(q(\theta), \theta) - c_q(q(\theta), \theta)]f(\theta)}{c_{q\theta}(q(\theta), \theta)} + 1 - F(\theta) = \frac{[u_q(q(\hat{\theta}), \hat{\theta}) - c_q(q(\hat{\theta}), \hat{\theta})]f(\hat{\theta})}{c_{q\theta}(q(\hat{\theta}), \hat{\theta})} + 1 - F(\hat{\theta}). \tag{C3}$$

Section 3.2 will give a shadow-value interpretation to the terms on both sides of (C3) and thereby provide some intuition for this condition.

### *3.2. Monotone solutions: Distortion and characterization*

This section gives a characterization of increasing solutions to problem (SB) and establishes that such solutions lie above $q^r$. That is, the distortion caused by binding non-local incentive constraints counteracts the usual distortion by local incentive constraints. Lemmas 3–5 show that only a certain subset of non-local incentive constraints can bind. This restriction makes it possible to identify implementable variations which are used to derive results on the optimal distortion (theorem 1). These results will be used later in the algorithm of section 4.[10]

[10] A more direct approach to the problem would be to utilize the generalized Kuhn-Tucker theorem; see e.g. [34, ch. 9.4]. However, the resulting first-order condition appears to be intractable.




(a) no upwards binding (b) no overlap

Figure 3: non-binding incentive constraints

This section assumes that the solution is monotone. Section 5 will give sufficient conditions for this to be the case. It should be pointed out that a strictly increasing solution has to be above $s$ as a result of the monotonicity condition in lemma 1.

**Corollary 1.** *Let $q$ be an implementable and strictly increasing decision function. Then $q(\theta) \geq s(\theta)$ for all $\theta \in \Theta \setminus \{\underline{\theta}\}$.*

Lemma 3 states that – in monotone solutions – no type will be indifferent between the contract designated for him and a contract that involves a higher decision. The intuition for lemma 3 is the same as in models with SC. Increasing the decision increases the costs for higher types less than for lower types. This is true as $c_{q\theta}(q', \theta) < 0$ for all $q' \geq q(\theta)$ in a monotone decision $q$, see figure 1b.

**Lemma 3.** *Let $q$ be an increasing decision function such that $q(\theta) \geq s(\theta)$ for all $\theta \in \Theta \setminus \{\underline{\theta}\}$. Then, $\Phi(\theta', \hat{\theta}') > 0$ for any two types $\theta'$, $\hat{\theta}' \in \Theta$ such that $\theta' < \hat{\theta}'$ and $q^+(\theta') < q(\hat{\theta}')$.*

**Proof.** Let $\hat{\theta}' > \theta'$. Then, $\Phi(\theta', \hat{\theta}') = -\int_{\theta'}^{\hat{\theta}'} \int_{q(x)}^{q(\hat{\theta}')} c_{q\theta}(y, x)\; dy\; dx > 0$ where the inequality follows from the following two observations: first, $q(\theta) \geq s(\theta)$ for all $\theta \in (\theta', \hat{\theta}')$ implies that the integrand is strictly negative for almost all arguments $(y, x)$. Second, monotonicity of $q$ implies $q(x) \leq q(\hat{\theta}')$ for all $x \in (\theta', \hat{\theta}')$ and $q(x) < q(\hat{\theta}')$ for $x$ close enough to $\theta'$ as $q^+(\theta') < q(\hat{\theta}')$. See figure 3a for a graphical representation. □

One of the results of this paper is that binding non-local incentive constraints can affect the solution without leading to irregularities, i.e. discontinuities or bunching. The following lemma shows that some irregularities can be ruled out on the grounds of incentive compatibility alone. Say (NLIC) holds with equality for $\theta$ and $\hat{\theta}$. The decision is then continuous at $\hat{\theta}$ (assuming that the monotonicity constraint does not bind for $\hat{\theta}$). Furthermore, the decision has to be strictly increasing in an open neighborhood of $\theta$ (assuming that the decision is continuous at $\theta$).

**Lemma 4.** *Let $q$ be an implementable and increasing decision function. Suppose $\Phi(\theta, \hat{\theta}) = 0$ for $\theta, \hat{\theta} \in \Theta$ such that $\theta > \hat{\theta}$ and $q(\theta) > q(\hat{\theta})$. If $\hat{\theta} > \underline{\theta}$ and there is no type $\tilde{\theta} \neq \hat{\theta}$ such that either $q(\tilde{\theta}) = q^-(\hat{\theta})$ or $q(\tilde{\theta}) = q^+(\hat{\theta})$, then $q$ is continuous at $\hat{\theta}$. If $q$ is continuous at $\theta$ and $\theta < \bar{\theta}$, then there does not exist a type $\theta' \neq \theta$ with $q(\theta') = q(\theta)$.*

The following lemma implies that binding non-local incentive constraints cannot "overlap". More precisely, if (NLIC) is satisfied with equality for $\theta$ and $\hat{\theta}$, then (NLIC) cannot hold with equality for a type in the interval $[\hat{\theta}, \theta]$ and a type outside this interval (abstracting from the trivial exception where an interval of types is assigned the same decision).

**Lemma 5.** *Let $q$ be an implementable and increasing decision function. Suppose $\Phi(\theta, \hat{\theta}) = 0$ for $\theta, \hat{\theta} \in \Theta$ such that $\theta > \hat{\theta}$. Then, $\Phi(\theta', \hat{\theta}') > 0$ for any pair of types $\theta', \hat{\theta}'$ such that $\hat{\theta}' \in (\hat{\theta}, \theta]$, $\theta' > \theta$ and there is no* *type $\tilde\theta \neq \hat\theta'$ such that $q(\tilde\theta) = q(\hat\theta')$. Furthermore, if there is no type $\tilde\theta \neq \hat\theta$ such that $q(\tilde\theta) = q(\hat\theta)$, $\Phi(\theta', \hat\theta') > 0$ for any pair of types $\theta'$, $\hat\theta'$ such that $\theta' \in [\hat\theta, \theta)$, $\hat\theta' < \hat\theta$.*

**Proof.** The proof is by contradiction. Suppose, contrary to the lemma, there are types $\theta_1 > \theta_2 \geq \theta_3 > \theta_4$ such that $\Phi(\theta_1, \theta_3) = \Phi(\theta_2, \theta_4) = 0$ and there is no type $\tilde\theta \neq \theta_3$ such that $q(\tilde\theta) = q(\theta_3)$. Then, the incentive constraint between $\theta_1$ and $\theta_4$ will be violated, i.e. $\Phi(\theta_1, \theta_4) < 0$:

$$\begin{aligned}
\Phi(\theta_1,\theta_4) &= -\int_{\theta_4}^{\theta_1}\int_{q(\theta_4)}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \\
&= -\int_{\theta_4}^{\theta_2}\int_{q(\theta_4)}^{q(x)} c_{q\theta}(y,x)\,dy\,dx - \int_{\theta_2}^{\theta_1}\int_{q(\theta_4)}^{q(\theta_3)} c_{q\theta}(y,x)\,dy\,dx - \int_{\theta_2}^{\theta_1}\int_{q(\theta_3)}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \\
&= -\int_{\theta_4}^{\theta_2}\int_{q(\theta_4)}^{q(x)} c_{q\theta}(y,x)\,dy\,dx - \int_{\theta_2}^{\theta_1}\int_{q(\theta_4)}^{q(\theta_3)} c_{q\theta}(y,x)\,dy\,dx \\
&\quad + \int_{\theta_3}^{\theta_2}\int_{q(\theta_3)}^{q(x)} c_{q\theta}(y,x)\,dy\,dx - \int_{\theta_3}^{\theta_1}\int_{q(\theta_3)}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \\
&= -\Phi(\theta_2,\theta_3) - \int_{\theta_2}^{\theta_1}\int_{q(\theta_4)}^{q(\theta_3)} c_{q\theta}(y,x)\,dy\,dx < 0
\end{aligned}$$

The second and third equality involve simple, splitting up the integral steps (and can be visualized in figure 3b) and the fourth uses the fact that $\Phi(\theta_1, \theta_3) = \Phi(\theta_2, \theta_4) = 0$. The last inequality follows from the incentive compatibility between $\theta_2$ and $\theta_3$ as well as the following idea: $\Phi(\theta_2, \theta_4) = 0$ and the fact that $\theta_2$ is interior imply $\int_{q(\theta_4)}^{q^-(\theta_2)} c_{q\theta}(y, \theta_2)\,dy \geq 0$ by (C1) (with equality if $q$ is continuous at $\theta_2$).

If $\theta_2 > \theta_3$ or if $q$ is left-continuous at $\theta_2$, the monotonicity of $q$ implies $q(\theta_3) \leq q^-(\theta_2)$ and therefore $\int_{q(\theta_4)}^{q(\theta_3)} c_{q\theta}(y, \theta_2)\,dy \geq 0$ (see figure 3b). The inequality above follows then from $c_{q\theta\theta} > 0$. If $\theta_2 = \theta_3$ and $q$ is not left-continuous at $\theta_2$, the first statement of lemma 4 implies (by $\Phi(\theta_1, \theta_2) = 0$) that for some $\varepsilon > 0$ either $q(\theta) = q^b$ for all $\theta \in (\theta_2, \theta_2 + \varepsilon]$ or $q(\theta) = q^b$ for all $\theta \in [\theta_2 - \varepsilon, \theta_2)$. Note that $q(\theta_2) < q^+(\theta_2)$ in the first case as there is no other type with the same decision as $\theta_3 = \theta_2$ by assumption. In both cases, $\int_{\theta_2}^{\theta_1} c_{q\theta}(q(\theta_2), x)\,dx \geq 0$ as otherwise $\Phi(\theta_1, \theta_2 + \varepsilon') < 0$ for some $\varepsilon' > 0$. But then $c_{qq\theta} < 0$ implies $\int_{\theta_2}^{\theta_1}\int_{q(\theta_4)}^{q(\theta_3)} c_{q\theta}(y, x)\,dy\,dx > 0$. □

As a special case, i.e. with $\hat\theta' = \theta$, the preceding lemma includes the following: if $\theta$ is indifferent between his and $\hat\theta$'s contract, then no other type $\theta'$ is indifferent between his contract and $\theta$'s contract, i.e. $\Phi(\theta', \theta) > 0$ for all $\theta' \in \Theta \setminus \{\theta\}$. The top panel of figure 4 summarizes the previous lemmas by showing how non-local incentive compatibility constraints could bind.

The previous three lemmas apply to any increasing (and implementable) decision function. The following results focus on the decision function that is optimal. Before stating the main result of this section, I want to clarify the terms used in the theorem. The theorem restricts itself to càdlàg functions.[11] For monotone solutions, this is, however, without loss of generality since a monotone function $q$ is continuous almost everywhere and both one-sided limits exist everywhere. Hence, using $q^+$ instead of $q$ will not change the principal's payoff and is incentive compatible.

**Theorem 1.** *Let $q \geq s$ be an increasing solution to problem (SB). Assume that $q$ is a càdlàg function. Then $q \geq q^r$ and*

$$[u_q(q(\theta), \theta) - c_q(q(\theta), \theta)]f(\theta) + (1 - F(\theta))c_{q\theta}(q(\theta), \theta) = \eta(\theta)c_{q\theta}(q(\theta), \theta) \tag{4}$$

*holds, where $\eta(\theta)$ is a non-negative, almost everywhere differentiable càdlàg function with the following properties:*

[11] A càdlàg function is defined as a function that is right continuous with left hand side limit at each point of the domain.

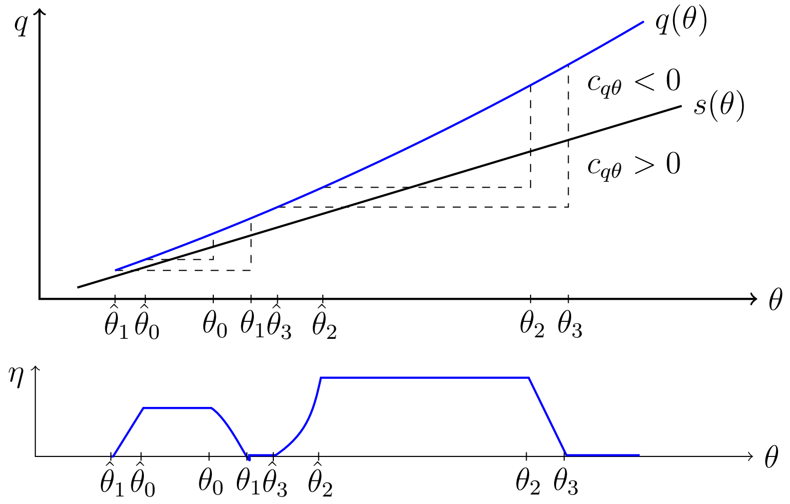



Figure 4: how incentive constraints can bind

- *Let* $(\theta_1, \theta_2)$ *be an interval such that* $q$ *is strictly increasing on* $(\theta_1, \theta_2)$ *and* $\Phi(\theta', \hat{\theta}') > 0$ *for every* $\hat{\theta}' \in (\theta_1, \theta_2)$ *and every* $\theta' \in \Theta \setminus \{\hat{\theta}'\}$*. Then,* $\eta$ *is decreasing on* $(\theta_1, \theta_2)$*.*
- *Let* $(\theta_1, \theta_2)$ *be an interval such that* $q$ *is strictly increasing on* $(\theta_1, \theta_2)$ *and* $\Phi(\theta', \hat{\theta}') > 0$ *for every* $\theta' \in (\theta_1, \theta_2)$ *and every* $\hat{\theta}' \in \Theta \setminus \{\theta'\}$*. Then,* $\eta$ *is increasing on* $(\theta_1, \theta_2)$*.*
- *Let* $\Phi(\theta', \hat{\theta}') = 0$ *and let* $q$ *be strictly increasing in open neighborhoods around* $\theta'$ *and* $\hat{\theta}'$*. Then,* $\eta$ *is decreasing in an open neighborhood of* $\theta'$*. Furthermore,* $\eta$ *is increasing in an open neighborhood of* $\hat{\theta}'$*.*
- $\eta^-(\bar{\theta})$ *is zero if* $\Phi(\bar{\theta}, \hat{\theta}) > 0$ *for all* $\hat{\theta} < \bar{\theta}$*.*
- $\eta^+(\underline{\theta})$ *is zero if* $\Phi(\theta, \underline{\theta}) > 0$ *for all* $\theta > \underline{\theta}$*.*
- *Let* $(\theta_1, \theta_2)$ *be an interval such that* $q$ *is constant on* $(\theta_1, \theta_2)$ *but* $q$ *is neither constant on* $(\theta_1 - \varepsilon, \theta_2)$ *nor on* $(\theta_1, \theta_2 + \varepsilon)$ *for any* $\varepsilon > 0$*. Then,* $\eta^+(\theta_1) \leq \eta^-(\theta_2)$*.*
- *Let* $\theta'$ *be a type such that* $q^-(\theta') < q^+(\theta')$*. Then,* $\eta^-(\theta') \geq \eta^+(\theta')$*.*

The theorem states that $q \geq q^r$. Put differently, distortion from binding non-local incentive constraints counteracts the usual downward distortion in (RP). The intuition for this result can be illustrated with figure 1b. (NLIC) is violated if the integral of $c_{q\theta}$ over the shaded area is positive. To satisfy incentive compatibility, the decision can be raised for all types between $\hat{\theta}$ and $\theta$. The inequality $c_{q\theta} < 0$ holds in the additional shaded area and the incentive problem is therefore mitigated.[12]

One noteworthy point is that (NLIC) is mainly relaxed by increasing the decision of types at which the incentive constraint is slack; i.e. if (NLIC) is violated for $\theta'$ and $\hat{\theta}'$, it is less $q(\theta')$ and $q(\hat{\theta}')$ that have to be increased but mainly $q$ for the types between $\hat{\theta}'$ and $\theta'$. To see the intuition, recall that $\pi_\theta(\theta) = -c_\theta(q(\theta), \theta)$ for almost all types and that $c_{q\theta}(q(\theta), \theta) < 0$. Increasing $q$ will therefore raise the slope of the rent function $\pi$. Consequently, an increased decision for types in $(\hat{\theta}', \theta')$ will increase the rent of $\theta'$ at his own contract which relaxes the non-local incentive constraint.

Note that $\eta$ can simply be defined by (4). The properties of $\eta$ are then derived by showing that $q$ could be changed in a way that is incentive compatible and increases the principal's payoff if these properties were

[12] In models with single crossing, the decision can be below $q^r$ for some types if the monotonicity constraint is violated under $q^r$. This does, however, not contradict the theorem because the assumptions of this paper imply that $q^r$ is strictly increasing (lemma 2).

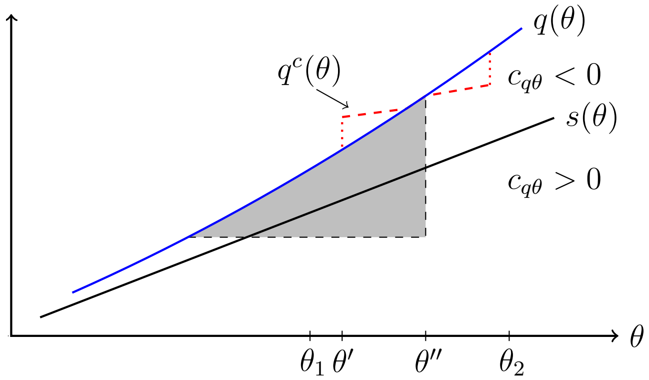



Figure 5: implementable variation

not satisfied. Figure 5 illustrates one changed decision that is used to prove the first bullet point. Such a variation turns out to (i) be implementable if $\Phi(\theta', \hat{\theta}') > 0$ for all $\hat{\theta}' \in (\theta_1, \theta_2)$ and $\theta' \neq \hat{\theta}'$ and (ii) increase the principal's payoff if $\eta$ was not decreasing on $(\theta_1, \theta_2)$.

The properties of $\eta$ have an intuitive interpretation. The left-hand side of (4) is well known from models with SC: Increasing $q(\theta)$ affects the surplus from type $\theta$ but also the rents of all types above $\theta$. Marginally increasing $q(\theta)$ will also relax all non-local incentive constraints from types $\theta' > \theta$ to types $\hat{\theta}' < \theta$ (see figure 1b). As these incentive constraints can be expressed as integrals over $c_{q\theta}$, see (NLIC), the "amount" by which those non-local incentive constraints are relaxed is given by $c_{q\theta}(q(\theta), \theta)$ which can be found on the right-hand side of (4). Consequently, $\eta(\theta)$ could be interpreted as the shadow value of all non-local incentive constraints from types $\theta' > \theta$ to types $\hat{\theta}' < \theta$.

Consider a type $\theta$ such that (NLIC) binds from $\theta$ to $\hat{\theta}$. Take a type slightly above $\theta$, say $\theta + \varepsilon$, and a type slightly below $\theta$, say $\theta - \varepsilon$. Because of lemmas 3 and 5, all incentive constraints relaxed by increasing $q(\theta + \varepsilon)$ are also relaxed by increasing $q(\theta - \varepsilon)$. The incentive constraint binding from $\theta$ is relaxed by increasing $q(\theta - \varepsilon)$ but not by increasing the decision of type $\theta + \varepsilon$. Hence, the shadow value of non-local incentive constraints has to be higher at $\theta - \varepsilon$ than at $\theta + \varepsilon$. This is in line with the third property of $\eta$ given in theorem 1: $\eta$ is decreasing in a neighborhood of $\theta$.

From this intuition, it seems that $\eta$ should be strictly lower for types in $(\theta, \theta + \varepsilon)$ than for types in $(\theta - \varepsilon, \theta)$ if (NLIC) binds from $\theta$ (in contrast to $\eta$ being "weakly" decreasing in an open neighborhood of $\theta$). Following the usual interpretation of shadow values, one can distinguish between types $\theta$ where merely $\Phi(\theta, \hat{\theta}) = 0$ and types from which a non-local incentive constraint binds in the sense that it affects the solution. I will say that *(NLIC) binds from* $\theta$ if $\Phi(\theta, \hat{\theta}) = 0$ for some $\hat{\theta} \neq \theta$ and either $\eta$ is strictly higher for types in $(\theta - \varepsilon, \theta)$ than for types in $(\theta, \theta + \varepsilon)$ for some $\varepsilon > 0$, or $\theta = \bar{\theta}$. If $\eta$ is constant (but not necessarily 0!) on an open interval of types, I will say that *(NLIC) is slack* for the types in this interval.

The shadow-value interpretation also provides some intuition for the necessary condition (C3) which states that $\eta(\theta) = \eta(\hat{\theta})$ when (NLIC) binds from $\theta$ to $\hat{\theta}$. This interpretation makes sense in light of lemma 5. Because there is no overlap in binding incentive constraints, the non-local incentive constraints relaxed by increasing $q(\theta)$ are the same as the ones relaxed by increasing $q(\hat{\theta})$. Consequently, the shadow value of relaxing those constraints is the same for the two types. The two properties for the boundary types are also straightforward; increasing the boundary types' decision does not affect non-local incentive constraints of other types.

Some of the properties of $\eta$ in theorem 1 hold only at types where the decision is strictly increasing. The reason is that – the way (4) is written – $\eta$ captures not only the effect of non-local incentive constraints but also the effect of the (local) monotonicity constraint. If one wants to avoid this cluttering of effects, it is straightforward to explicitly introduce the monotonicity constraint. This is done in section WA 2.3 of the webappendix.

*3.3. Distortion at the top*

The idea underlying distortion in models with SC is that downward distortion at a certain type allows the principal to reduce the rent of all higher types. As one moves across types towards $\bar{\theta}$, there are fewer and fewer higher types and therefore the distortion vanishes. Eventually, there is no distortion for $\bar{\theta}$. This is the famous "no distortion at the top" result.

Now consider the case where the non-local incentive constraint binds from $\bar{\theta}$ to a mass of types $\hat{\theta}$ (or to $\underline{\theta}$). Then the shadow value $\eta(\theta)$ will be strictly positive and bounded away from 0 for types slightly below $\bar{\theta}$. Hence, these types have a decision $q(\theta)$ which is at least $\varepsilon$ above their relaxed decision $q^r(\theta)$ for some $\varepsilon > 0$. The same must then apply to $\bar{\theta}$ because of the monotonicity constraint. Put differently, $\eta(\bar{\theta}) > 0$ and therefore $q(\bar{\theta}) > q^{fb}(\bar{\theta})$; there is distortion at the top.

The main difference to the "no distortion at the top" intuition is the following. The effect of distortion on the binding non-local incentive constraint does not necessarily vanish as we move across types towards $\bar{\theta}$. Increasing the decision of any type below $\bar{\theta}$ will – through the slope of the rent function – increase the rent of $\bar{\theta}$ and therefore relax the binding non-local incentive constraints. Note that the distortion created by (NLIC) is upwards, i.e. there is no downward distortion at the top type as $q \geq q^r$ (theorem 1).

A simple sufficient condition for "no distortion at the top" can be formulated using (C1):

$$\int_{q^r(\underline{\theta})}^{q^{fb}(\bar{\theta})} c_{q\theta}(q, \bar{\theta})\, dq \leq 0 \tag{5}$$

is sufficient for $q(\bar{\theta}) = q^{fb}(\bar{\theta})$. The reason is that (C1) cannot hold with inequality in this case as $q(\bar{\theta}) \geq q^r(\bar{\theta}) = q^{fb}(\bar{\theta})$ and $q(\hat{\theta}) \geq q^r(\hat{\theta}) \geq q^r(\underline{\theta})$ by theorem 1.

## 4. Continuous and strictly monotone solutions

This section introduces an algorithm that determines continuous and strictly increasing solutions. The algorithm is based on a necessary condition satisfied by such solutions which is established first.

For this section, I assume that the solution satisfies three conditions: (i) continuity; (ii) strict monotonicity; (iii) for a given type $\theta$ and a given $\eta \in [0, \max \eta(\theta)]$

$$\text{there is a unique } q \geq q^r(\theta) \text{ solving } [u_q(q,\theta) - c_q(q,\theta)]f(\theta) + (1 - F(\theta) - \eta)c_{q\theta}(q,\theta) = 0 \tag{UQ}$$

(where (UQ) is basically (4) rewritten). This unique $q$ is denoted by $q(\theta, \eta)$ (with slight abuse of notation). Note that by assumption 1, (UQ) is automatically satisfied for $\eta \leq 1 - F(\theta)$. Furthermore, (UQ) and assumption 1 imply that $q(\theta, \eta)$ is strictly increasing in $\eta$, i.e. a higher shadow values correspond to a higher decision. The next section will present sufficient conditions on the primitives of the problem that ensure that these three conditions hold.

The rough idea behind theorem 2 is the following. Take the optimal contract $q$ as given and say (NLIC) binds from $\theta'$ to $\hat{\theta}'$. Then $\Phi(\theta', \hat{\theta}') = 0$ and $(\theta', \hat{\theta}')$ minimize $\Phi$ which implies $\Phi_\theta(\theta, \hat{\theta}') \leq 0$ for $\theta \in (\theta' - \varepsilon, \theta')$ and $\Phi_\theta(\theta, \hat{\theta}') \geq 0$ for $\theta \in (\theta', \theta' + \varepsilon)$ for some $\varepsilon > 0$. Let both $\theta'$ and $\hat{\theta}'$ have the same $\eta$, call it $\eta'$, which is implied by (C3) if both types are interior. I will now argue that $(\theta', \hat{\theta}')$ still minimizes $\Phi$ if the optimal decision $q$ is manipulated as follows. According to theorem 1, $\eta$ is decreasing in an open neighborhood of $\theta'$. Now manipulate $q$ such that $\eta$ defined by (4) is constant (and equal to $\eta'$) in some neighborhood of $\theta'$; i.e. assign $q(\theta, \eta')$ instead of $q(\theta)$ to types in this neighborhood. This implies that the decision of types slightly below $\theta'$ is decreased and the decision of types slightly above $\theta'$ is increased. This manipulation decreases $\Phi_\theta(\theta, \hat{\theta}')$ for $\theta \in (\theta' - \varepsilon, \theta')$, increases $\Phi_\theta(\theta, \hat{\theta}')$ for types $\theta \in (\theta', \theta' + \varepsilon)$ and leaves $\Phi_\theta(\theta, \hat{\theta}')$ zero for $\theta = \theta'$. Consequently, $(\theta', \hat{\theta}')$ is a local minimizer of this manipulated $\Phi$. Theorem 2 shows that this is still the case if we manipulate $q$ not only in a neighborhood of $\theta$ but for all types, namely if every type $\theta$ is assigned $q(\theta, \eta')$. The theorem states, for this case, even that $(\theta', \hat{\theta}')$ minimize the resulting $\Phi$ not only locally but also on the interval $[\hat{\theta}' - \varepsilon, \theta' + \varepsilon]$. I use the following notation in the remainder of the paper: $(\theta(\eta), \hat{\theta}(\eta))$ denotes the minimizer of $\Phi^\eta(\theta, \hat{\theta})$ which is defined as $\Phi(\theta, \hat{\theta})$ under $q(\theta, \eta)$:

$$\Phi^\eta(\theta, \hat{\theta}) = -\int_{\hat{\theta}}^{\theta} \int_{q(\hat{\theta},\eta)}^{q(x,\eta)} c_{q\theta}(y, x)\, dy\, dx.$$

**Theorem 2.** *Let q solve problem (SB). Assume that (NLIC) binds from $\theta'$ to $\hat{\theta}'$.*[13] *Then, $(\theta', \hat{\theta}')$ minimizes $\Phi^{\eta'}(\theta, \hat{\theta})$ on $[\hat{\theta}' - \varepsilon, \theta' + \varepsilon]$ for some $\varepsilon > 0$ where $\eta' = min(\eta(\theta'), \eta(\hat{\theta}'))$.*[14] *Furthermore, $\Phi^{\eta'}(\theta', \hat{\theta}') \leq 0$.*

The theorem says that types for which (NLIC) binds under the optimal decision are minimizers of their respective $\Phi^\eta$. The inequality condition is useful as it can help to determine the maximum of $\eta(\theta)$, which I will call $\bar{\eta}$, in the solution: because of lemma 5 and theorem 1, (NLIC) will not bind to or from types in $(\hat{\theta}(\bar{\eta}), \theta(\bar{\eta}))$ which means that $\eta$ is constant on $(\hat{\theta}(\bar{\eta}), \theta(\bar{\eta}))$. Therefore, $\Phi(\theta(\bar{\eta}), \hat{\theta}(\bar{\eta})) = \Phi^{\bar{\eta}}(\theta(\bar{\eta}), \hat{\theta}(\bar{\eta}))$ and incentive compatibility requires then that the inequality in the theorem holds with equality for $\eta' = \bar{\eta}$.

For a given $\eta$, $\Phi^\eta$ and its minimizers can be determined without knowing the optimal decision. Suppose there is – for each $\eta \geq 0$ – only one local minimizer of $\Phi^\eta$ and that the condition above determines $\bar{\eta}$. Then the solution can be constructed as follows. For $\eta \in [0, \bar{\eta}]$, the optimal decision for the $\Phi^\eta$ minimizing types is their respective $q(\theta, \eta)$. The optimal decision for those types at which (NLIC) is slack is then determined by (4) as theorem 1 implies that $\eta$ is constant on intervals of such types (and therefore determined by the closest type from/to which (NLIC) binds).

Assume now that $\Phi^\eta$ has only one local minimizer $(\theta(\eta), \hat{\theta}(\eta))$ for every $\eta \in [0, \bar{\eta}]$ where $\bar{\eta}$ can be obtained through the condition $\Phi^{\bar{\eta}}(\theta(\bar{\eta}), \hat{\theta}(\bar{\eta})) = 0$. The case of several minimizers is briefly discussed at the end of this section and more extensively in the webappendix. The idea above leads then to the following algorithm:

1. Solve (4) for $q(\theta, \eta)$ (assuming $\eta \geq 0$) and calculate $\Phi^\eta(\theta, \hat{\theta})$.
2. Minimize $\Phi^\eta(\theta, \hat{\theta})$ over $\theta$ and $\hat{\theta}$.[15] Denote the minimizer by $(\theta(\eta), \hat{\theta}(\eta))$.
3. Define $\bar{\eta}$ as the $\eta$ solving $\Phi^\eta(\theta(\eta), \hat{\theta}(\eta)) = 0$.
4. The solution is then the following:
   (a) types from/to which non-local incentive constraints are binding: for all $\eta \in [0, \bar{\eta}]$, $q(\theta(\eta)) = q(\theta(\eta), \eta)$ and $q(\hat{\theta}(\eta)) = q(\hat{\theta}(\eta), \eta)$ (and therefore $\eta(\theta(\eta)) = \eta$ and $\eta(\hat{\theta}(\eta)) = \eta$).[16]
   (b) types with $\eta = 0$: for types above $\theta(0)$ and below $\hat{\theta}(0)$, the optimal decision is the relaxed decision.[17]
   (c) types where non-local incentive constraints are slack but $\eta > 0$: let $\tilde{\Theta}$ be the set of all types that were assigned a decision in (a) or (b). For $\theta \notin \tilde{\Theta}$, define $\tilde{\theta}(\theta) = \min \tilde{\Theta}$ s.t.: $\tilde{\theta}(\theta) \geq \theta$. Then $\eta(\theta) = \eta(\tilde{\theta}(\theta))$ for $\theta \notin \tilde{\Theta}$. The optimal decision for these types is $q(\theta, \eta(\theta))$.

The algorithm will be illustrated with an example in section 6.

**Proposition 1.** *Let the solution to problem (SB) be (i) continuous, (ii) strictly increasing and (iii) satisfy (UQ). Let $\Phi^\eta$ have a unique local minimizer for $\eta \in [0, \bar{\eta}]$. Then the outcome of the algorithm solves problem (SB).*

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[13] I use "binding" in the sense explained in section 3.2.

[14] If $\theta'$ and $\hat{\theta}'$ are interior, (C3) implies $\eta' = \eta(\theta') = \eta(\hat{\theta}')$.

[15] The domain of the minimization is $[\underline{\theta}, \bar{\theta}]$. It will be pointed out later that the domain is $[\underline{\theta}, \theta'(\eta)]$ where $\theta'(\eta)$ is defined by $1 - F(\theta') = \eta$ if the sufficient conditions of proposition 4 are satisfied.

[16] The boundary types $\bar{\theta}$ and $\underline{\theta}$ can minimize $\Phi^\eta$ for several values of $\eta$. In this case, $q(\bar{\theta}) = q(\bar{\theta}, \eta')$ (and $q(\underline{\theta}) = q(\underline{\theta}, \eta')$ respectively) where $\eta'$ is the highest $\eta$ for which $\bar{\theta} = \theta(\eta)$ (and $\underline{\theta} = \hat{\theta}(\eta)$ respectively).

[17] This is not in conflict with (a) as $\theta(\eta)$ is increasing and $\hat{\theta}(\eta)$ is decreasing in $\eta$; see the proof of proposition 1.

It was assumed that there was a unique local minimizer of $\Phi^\eta$. If instead there are several $\Phi^\eta$ minimizers $(\theta_i(\eta), \hat{\theta}_i(\eta))$ for a given $\eta$, theorem 2 alone does not necessarily determine for which of these minimizers (NLIC) binds. A variation of the algorithm can still be used if it can be determined for which minimizers (NLIC) binds and for which (NLIC) does not bind. While this is a hard problem in general, similar techniques as in the proof of theorem 2, i.e. utilizing the one-to-one relationship between $q$ and $\eta$ and incentive compatibility, can be used in special cases. The webappendix contains several such results for specific configurations of $\Phi^\eta$ minimizers, e.g. nested minimizers and overlapping minimizers.

## 5. Sufficient conditions for continuous, monotone contract menus

The previous section showed how to derive the solution assuming that this solution existed, was continuous, strictly increasing and satisfied (UQ). This section gives sufficient conditions under which these conditions are satisfied. Hence, before using the algorithm of the previous section, the sufficient conditions of this section should be verified.

### *5.1. Existence and monotonicity*

The assumptions in section 2 alone cannot rule out that $q$ discontinuously jumps across $s$. One technical condition has to be added. Given this condition, assumption 2 ensures that the solution is monotone (and not inversely U-shaped).

To state this technical condition, two associated decisions have to be defined for each decision $y \in [0, s(\theta)]$. First, $q^s(y,\theta) \geq s(\theta)$ is defined by $c_\theta(y,\theta) = c_\theta(q^s(y,\theta),\theta)$; i.e. $q^s(y,\theta)$ is the decision that would lead to the same slope of the rent function $\pi(\theta)$ as $y$ at type $\theta$. Second, $q^v(y,\theta) \geq s(\theta)$ is defined by $[u(y,\theta) - c(y,\theta)]f(\theta) + (1 - F(\theta))c_\theta(y,\theta) = [u(q^v(y,\theta),\theta) - c(q^v(y,\theta),\theta)]f(\theta) + (1 - F(\theta))c_\theta(q^v(y,\theta),\theta)$, i.e. $q^v(y\theta)$ is the decision which gives the same virtual valuation as $y$. Since $c_\theta(q,\theta)$ and (RP) are strictly concave, the two associated decisions $q^s$ and $q^v$ are well defined. By the strict concavity of $c_\theta$ and (RP), the condition below ensures that the principal's objective is higher if he assigns $q^s(y,\theta) > s(\theta)$ instead of $y < s(\theta)$ to type $\theta$.

**Proposition 2.** *Assume $q^v(y,\theta) \geq q^s(y,\theta)$ for all $y \in [0, s(\theta)]$ and all $\theta \in \Theta$. Then a solution $q \geq s$ to problem (SB) exists and $q$ is increasing.*

Note that the imposed condition is automatically satisfied for $y$ close to $s(\theta)$ as $q^r(\theta) > s(\theta)$ by lemma 2. To illustrate, take the example from section 1.1 and assume that $u(q,\theta) = Sq$ with $S > 0$. Straightforward calculation shows that in this case $q^v(y,\theta) = 2q^r(\theta) - y$ and $q^s(y,\theta) = \theta^2 - y = 2s(\theta) - y$. Hence, $q^v(y,\theta) > q^s(y,\theta)$ by lemma 2 and – regardless of the type distribution – the conditions of the proposition are satisfied.

### *5.2. Continuity*

This subsection gives two sets of sufficient conditions for (i) the continuity of the solution, (ii) strict monotonicity and (iii) condition (UQ). The first sufficient condition ensures that $q(\theta,\eta)$ is strictly increasing in both its arguments which turns out to be sufficient for continuity and strict monotonicity of the optimal decision.

**Proposition 3.** *A monotone solution $q$ to problem (SB) is continuous and satisfies (UQ) if*

$$\frac{u_{qq}(x,\theta) - c_{qq}(x,\theta)}{c_{qq\theta}(x,\theta)} > \frac{u_q(x,\theta) - c_q(x,\theta)}{c_{q\theta}(x,\theta)} \tag{CVR}$$

*holds for all types and all $x \geq q^{fb}(\theta)$. If (CVR) holds and $q(\theta,\eta)$ is strictly increasing in type for any given $\eta \in [0,\bar{\eta}]$, then $q$ is strictly increasing.*[18]

[18] (CVR) has actually only to hold for $x \in [q^{fb}(\theta), \bar{q}]$ where $\bar{q}$ is defined as in the webappendix (section WA 2.1). Also note that (CVR) is automatically satisfied for $x \in (s(\theta), q^{fb}(\theta))$ because the left hand side of (CVR) is positive and the right hand side is negative for these $x$. As before, $\bar{\eta}$ is defined by $\min_{\theta,\hat{\theta}} \Phi^\eta(\theta,\hat{\theta}) = 0$.

To illustrate (CVR), take the cost function in the example in section 1.1 and assume that $u(q,\theta) = Sq$. It turns out that (CVR) is equivalent to the condition for $q^{fb}(\theta) > s(\theta)$, i.e. $S > 2\bar{\theta}$.

The following proposition provides a condition under which the solution is below the first-best decision. Having a solution below first best turns out to be sufficient for continuity and strict monotonicity of the solution. Furthermore, the property also holds locally. That is, if the decision is below $q^{fb}$ on some interval $(\theta_1, \theta_2)$, then the decision will be strictly increasing and continuous on $(\theta_1, \theta_2)$.

Before stating the proposition some additional notation is needed. Define $q^m(\theta) < s(\theta)$ such that $c_\theta(q^{fb}(\theta), \theta) = c_\theta(q^m(\theta), \theta)$, i.e. $q^{fb}(\theta) = q^s(q^m(\theta), \theta)$. Hence, $q^m$ is the decision function below the sign switch decision $s$ which would lead to the same slope of the rent function as $q^{fb}$. If $c_\theta(q^{fb}(\theta), \theta) < c_\theta(q, \theta)$ for all $q \in [0, s(\theta)]$, let $q^m(\theta)$ be 0. By $c_{qq\theta} < 0$, $q^m$ is well defined.

**Proposition 4.** *Let $q$ be an increasing solution to problem (SB) and let $\theta$ be a type such that $q^+(\theta) \leq q^{fb}(\theta)$. Then, $q$ is continuous at $\theta$. $q$ is strictly increasing on every interval $(\theta_1, \theta_2)$ such that $q(\theta) \leq q^{fb}(\theta)$ for all $\theta \in (\theta_1, \theta_2)$. Assume that $q^m$ is increasing and that there is no distortion at the top.*[19] *Then, $q \leq q^{fb}$ and therefore $q$ is continuous and strictly increasing.*

One example for a class of function where $q^m$ is increasing are cost functions of the form $c(q,\theta) = \theta q + \phi(q - \alpha\theta) + \gamma(\theta)$ where $\phi(\cdot)$ is a function of which the first three derivatives are positive. The interpretation of this cost function is that there is a "normal scale" of $\alpha\theta$. Producing above this normal scale is increasingly costly. Type reflects a trade-off between the size of the normal scale and marginal cost when producing within the normal scale. With such a cost function, any increasing and concave benefit function $u$ with $u_{q\theta} = 0$ and $q^{fb} > s$ yields an increasing $q^m$.

Although condition (UQ) is not necessarily satisfied given that $q \leq q^{fb}$, all results of section 4 still go through for the following reason. Knowing that $q \leq q^{fb}$ is equivalent to knowing that $\eta \leq 1 - F$. Given $\eta(\theta) \leq 1 - F(\theta)$, however, (4) defines a strictly monotone one-to-one relationship between $q(\theta)$ and $\eta(\theta)$ for a given type $\theta$. Guaranteeing this monotone one-to-one relationship was, of course, exactly what (UQ) was needed for in section 4. The only change in the algorithm is that in step 2 the domain of the minimization is $[\underline{\theta}, , \theta'(\eta)]$ where $\theta'(\eta)$ is defined by $1 - F(\theta') = \eta$; i.e. by $\eta(\theta) \leq 1 - F(\theta)$, it is clear that (NLIC) cannot bind from any type above $\theta'(\eta)$ for a given $\eta \geq 0$.

## 6. Numerical example and distortion for all types

I want to illustrate the algorithm with a numerical example which will also yield an interesting possibility result. Amend the example of section 1.1 by assuming $c(q,\theta) = q\theta + q^2/(2\theta) - \theta/10$, $u(q,\theta) = 21q\theta/10$ and let types be distributed uniformly between 1/4 and 1/2.

Hence, $s(\theta) = \theta^2$ and the first-best decision is $q^{fb}(\theta) = 11\theta^2/10$. The relaxed solution is $q^r(\theta) = \theta^2 + \theta^3/5$. The example is illustrated in figure 6. As the first-best decision is close to $s$ for all types, one would expect distortion above first best and (NLIC) to bind from the top to the bottom type. Indeed this is the case as shown below. According to proposition 2, the solution is monotone because $q^v(q,\theta) = -q + 2\theta^2 + 2\theta^3/5 > -q + 2\theta^2 = q^s(q,\theta)$.

Also (CVR) is satisfied as it boils down to

$$\frac{\frac{-1}{\theta}}{\frac{-1}{\theta^2}} > \frac{\frac{11}{10}\theta - \frac{q}{\theta}}{1 - \frac{q}{\theta^2}} \qquad \Leftrightarrow \qquad 0 > \frac{\frac{1}{10}\theta^2}{\theta^2 - q}$$

which holds for all $q > s(\theta) = \theta^2$. Hence, the solution is continuous. Next use the algorithm to compute a solution:

1. Solving (4) for $q$ yields
$$q(\theta,\eta) = \theta^2 + \frac{2\theta^3}{5(2-\eta)}$$

[19] Recall that (5) is a sufficient condition for the latter.

which is strictly increasing in type for $\eta < 2$. Computing $\Phi^\eta$ is slightly cumbersome but possible:

$$\begin{aligned}\Phi^\eta(\theta,\hat\theta) &= \int_{\hat\theta}^{\theta}\Big(c_\theta(q(\hat\theta,\eta),x) - c_\theta(q(x,\eta),x)\Big)\,dx\\ &= \hat\theta^2\left(\theta-\frac{4}{3}\hat\theta\right)+\frac{2\hat\theta^3(\theta-2\hat\theta)}{5(2-\eta)}+\frac{\hat\theta^4+\frac{4\hat\theta^5}{5(2-\eta)}+\frac{4\hat\theta^6}{25(2-\eta)^2}}{2\theta}-\frac{\theta^3}{6}+\frac{2\theta^5-12\hat\theta^5}{125(2-\eta)^2}.\end{aligned}$$

2. The first-order conditions for minimization, i.e. the equivalents of (C1) and (C2), are

$$\hat\theta^2+\frac{2\hat\theta^3}{5(2-\eta)}-\frac{1}{2\theta^2}\left(\hat\theta^4+\frac{4\hat\theta^5}{5(2-\eta)}+\frac{4\hat\theta^6}{25(2-\eta)^2}\right)\leq\frac{\theta^2}{2}-\frac{2\theta^4}{25(2-\eta)^2}\quad\text{with “=” if }\theta<\bar\theta,\tag{6}$$

$$\theta+\frac{1}{\theta}\left(\hat\theta^2+\frac{2\hat\theta^3}{5(2-\eta)}\right)\geq 2\hat\theta+\frac{2\hat\theta^2}{5(2-\eta)}\quad\text{with “=” if }\hat\theta>\underline{\theta}.\tag{7}$$

It is easy to verify that the first-order conditions are satisfied for the boundary types $\bar\theta = 1/2$ and $\underline{\theta} = 1/4$ for all $\eta \leq 1.7$. It is a bit more tedious to verify that $(\bar\theta, \underline{\theta})$ is the unique local minimizer (for $\eta \leq 1.7$), i.e. that there are no other local minimizers. This is done in the webappendix. For $\eta > 1.7$, minimizers are interior but those minimizers are not needed in the following; so I do not derive them here.

3. For $\bar\eta \approx 1.67$, $\Phi^{\bar\eta}(\bar\theta, \underline{\theta}) = 0$. Hence, (NLIC) will bind only from $\bar\theta$ to $\underline{\theta}$.
4. For all types, $\eta(\theta) = 1.67$. Hence,

$$q(\theta)=\theta^2+\frac{2\theta^3}{5(2-1.67)}.$$

Since $\eta(\theta) > 1 \geq 1 - F(\theta)$ for all $\theta$, all types have decisions above first best in the example. This result differs from results in the literature on type-dependent participation constraints such as Jullien [23]. In this literature, the decision can also be distorted above as well as below first best but there is always at least one type whose decision is undistorted. In the inverse U-shape solutions of Araujo and Moreira [5] there is also always one type with an undistorted decision.

**Theorem 3.** *The solution $q$ of problem (SB) can be such that $q(\theta) > q^{fb}(\theta)$ for all $\theta \in \Theta$.*

## 7. Conclusion

The paper characterizes monotone solutions in a screening environment where SC is violated. This implies that non-local incentive constraints can be binding. The distortion caused by non-locally binding incentive constraints counteract the normal rent extraction distortion. The solution can therefore be partly above as well as below the first-best decision. If a non-local incentive constraint binds from the highest type to the lowest type, the decision of all types can be distorted upward.

To conclude, I discuss relaxing the assumptions, possible generalizations and applications. First, for the results in section 3, the assumptions on $u_{q\theta}$, $u_{qq}$, $c_{qq}$ and the monotone hazard rate assumption in assumption 1 can be relaxed as long as the relaxed program remains concave and the relaxed decision remains strictly increasing; i.e. as long as the properties in lemma 2 hold. These properties of the relaxed program are used in deriving theorem 1. The assumption that $u_{qq} - c_{qq} \leq 0$ and the monotone hazard rate assumption are only used in the proof of proposition 4. The monotonicity properties of $\eta$ in theorem 1 hold more generally. The only requirement is that the variations in the proof (as for example indicated in figure 5) are implementable. Implementability follows from lemmas 3– 5 which only require that the third derivatives $c_{qq\theta}$ and $c_{q\theta\theta}$ do not change sign.

This leads to the question of the importance of these assumptions on third derivatives of $c$. It is not important which signs these third derivatives have as long as they do not switch sign which is illustrated

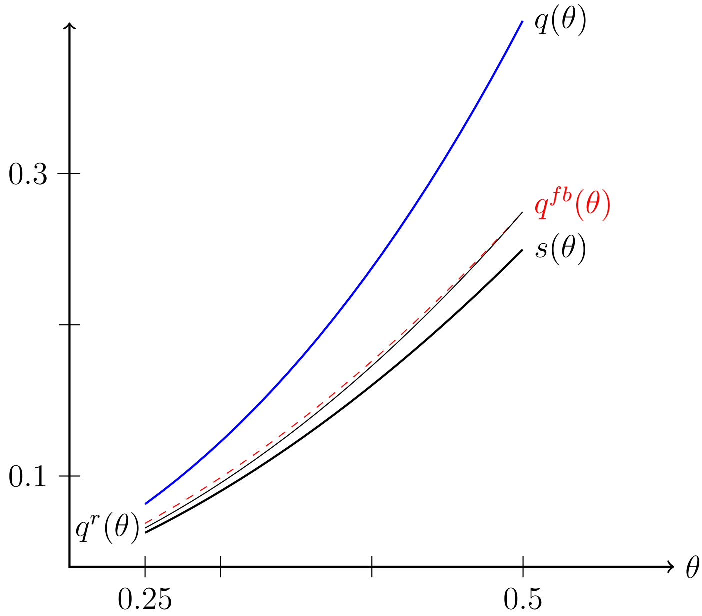


Figure 6: numerical example: all types are distorted

with an example in the webappendix. However, the assumption that they do not switch sign is necessary to make the analysis tractable. For illustration, suppose that the third derivatives of $c$ can switch sign in the relevant range of $q$ and $\theta$. In this case, there could be a second function $s_2(\theta)$ where $c_{q\theta}$ changes sign. Consider figure 3a and imagine such a function $s_2$ somewhere above $q$ and going through the shaded area. It is clear that in this case upwards-binding incentive constraints can no longer be ruled out, i.e. lemma 3 no longer holds (also lemma 5 is no longer valid). Nevertheless, theorem 1 will not become irrelevant in such a framework. As long as the changed decisions (as in figure 5) are implementable, theorem 1 holds. In particular, $\eta$ has to be constant on an interval of types for which all non-local incentive constraints are slack. Similarly, $\eta$ has to be decreasing in a neighborhood of a type $\theta$ satisfying (i) $\Phi(\theta, \hat{\theta}) = 0$ for some $\hat{\theta} < \theta$ and (ii) $\Phi(\theta', \theta) > 0$ for all $\theta' \neq \theta$.

Possible applications can be found in various fields of economics. While the paper uses the notation of a regulation or procurement setting, the same model is applicable, for example, in models of labor, insurance or monopoly pricing. It is hard to judge how common distortions above first best are in these applications. However, there is some evidence from non-linear pricing in telecommunication: consumers often buy packages where an additional unit of calling (or internet use) is for free. If the marginal costs of the provider are only $\varepsilon$ above zero, such a price scheme will lead to consumption above the socially optimal level.

The paper derives a new necessary condition that is based on minimizing the incentive compatibility constraint for a given shadow value. This condition is sufficient to calculate solutions that are continuous and strictly monotone. Future research will determine to what extent this condition can be generalized. It might be particularly interesting to see whether a similar procedure can be used in multidimensional screening models. More straightforward applications include areas in which violations of single crossing and binding non-local incentive constraints emerge naturally. This has been documented in common-agency games [25], provision of pre-sale information [28] and dynamic mechanism design problems with serially correlated types [35].

**Acknowledgments**

I want to thank Jan Boone, Bruno Jullien, Peter Norman Sørensen and Bert Willems for interesting discussions and numerous suggestions. I also want to thank two anonymous referees and the associate editor for their excellent suggestions. I have benefited from comments by Cédric Argenton, Johan Lagerlöf, Matthias Lang, Humberto Moreira, Jens Prüfer, François Salanié, Florian Schütt, Eric van Damme, Nick Vikander and seminar participants at the Toulouse School of Economics, University of Southern Denmark Odense, Aalto University Helsinki, University of Copenhagen, Lund University, UT Sydney, Monash University Melbourne, Adelaide University and Tilburg University as well as conference participants at EARIE, the ENTER Jamboree and the European Winter Meeting of the Econometric Society.

**Appendix**

**Proof of lemma 1 and 2:** see webappendix

**Proof of corollary 1:** Suppose there was a type $\theta' > \underline{\theta}$ such that $q(\theta') < s(\theta')$. Monotonicity of $q$ and continuity of $s$ then imply $q(\theta) < s(\theta)$ for all $\theta \in (\theta' - \varepsilon, \theta')$ for some $\varepsilon > 0$. As $q$ is increasing, $q$ is continuous almost everywhere. Hence, there exists a $\theta'' \in (\theta' - \varepsilon, \theta')$ such that $q$ is continuous at $\theta''$. The monotonicity condition of lemma 1 is violated at $\theta''$ contradicting that $q$ is implementable. □

**Proof of lemma 4:** Let $\Phi(\theta, \hat{\theta}) = 0$. First, it is shown that there cannot be a discontinuity at $\hat{\theta}$ under the conditions of the lemma. By way of contradiction, suppose that $q$ is discontinuous at $\hat{\theta}$, i.e. $q^-(\hat{\theta}) < q^+(\hat{\theta})$. As $q$ is increasing, either (i) $q(\hat{\theta}) = q^-(\hat{\theta})$ or (ii) $q(\hat{\theta}) = q^+(\hat{\theta})$ or (iii) $q(\hat{\theta}) \in (q^-(\hat{\theta}), q^+(\hat{\theta}))$.

In case (i), it must hold that $\int_{\hat{\theta}}^{\theta} c_{q\theta}(q^-(\hat{\theta}), x)\, dx \leq 0$ which is just (C2) adapted to apply for a right hand side discontinuity; i.e. if this did not hold, $\Phi(\theta, \hat{\theta} - \varepsilon) < 0$ for $\varepsilon > 0$ small enough which violates (NLIC). But then $\int_{\hat{\theta}}^{\theta} \int_{q^-(\hat{\theta})}^{q^+(\hat{\theta})} c_{q\theta}(y, x)\, dy\, dx < 0$ from $c_{qq\theta} < 0$. Hence, $\Phi(\theta, \hat{\theta}) + \int_{\hat{\theta}}^{\theta} \int_{q^-(\hat{\theta})}^{q^+(\hat{\theta})} c_{q\theta}(y, x)\, dy\, dx < 0$ as $\Phi(\theta, \hat{\theta}) = 0$ by assumption. This implies $\Phi(\theta, \hat{\theta} + \varepsilon) < 0$ for $\varepsilon > 0$ small enough as $\lim_{\varepsilon \to 0^+} \Phi(\theta, \hat{\theta} + \varepsilon) = \Phi(\theta, \hat{\theta}) + \int_{\hat{\theta}}^{\theta} \int_{q^-(\hat{\theta})}^{q^+(\hat{\theta})} c_{q\theta}(y, x)\, dy\, dx$, i.e. incentive compatibility is violated from $\theta$ to types slightly above $\hat{\theta}$. This is the desired contradiction.

In case (ii), it must hold that $\int_{\hat{\theta}}^{\theta} c_{q\theta}(q^+(\hat{\theta}), x)\, dx \geq 0$. But then $\int_{\hat{\theta}}^{\theta} \int_{q^-(\hat{\theta})}^{q^+(\hat{\theta})} c_{q\theta}(y, x)\, dy\, dx > 0$ from $c_{qq\theta} < 0$. Consequently, $\lim_{\varepsilon \to 0^+} \Phi(\theta, \hat{\theta} - \varepsilon) = \Phi(\theta, \hat{\theta}) - \int_{\hat{\theta}}^{\theta} \int_{q^-(\hat{\theta})}^{q^+(\hat{\theta})} c_{q\theta}(y, x)\, dy\, dx < 0$ and therefore incentive compatibility is violated from $\theta$ to types slightly below $\hat{\theta}$.

In case (iii), the same arguments as in case (i) apply if $\int_{\hat{\theta}}^{\theta} c_{q\theta}(q(\hat{\theta}), x)\, dx \leq 0$ while the same arguments as in case (ii) apply if $\int_{\hat{\theta}}^{\theta} c_{q\theta}(q(\hat{\theta}), x)\, dx > 0$.

Second, it is shown that $\theta < \bar{\theta}$ cannot be assigned the same decision as a type $\theta'$ if $q$ is continuous at $\theta$. Suppose, by way of contradiction, $q(\theta) = q(\theta')$ for some type $\theta' \neq \theta$. Note that, since $q$ is increasing, $q(x) = q(\theta) = q(\theta')$ for all $x \in [\min\{\theta, \theta'\}, \max\{\theta, \theta'\}]$. First, let $\theta < \theta'$. But then $\Phi(\theta', \hat{\theta}) < 0$ and (NLIC) is violated as

$$\Phi(\theta', \hat{\theta}) = -\int_{\hat{\theta}}^{\theta'} \int_{q(\hat{\theta})}^{q(x)} c_{q\theta}(y, x)\ dy\, dx = \Phi(\theta, \hat{\theta}) - \int_{\theta}^{\theta'} \int_{q(\hat{\theta})}^{q(\theta)} c_{q\theta}(y, x)\ dy\, dx < 0$$

where the last inequality follows from (C1) and $c_{q\theta\theta} > 0$.

Next, let $\theta > \theta'$. From condition (C1) and $c_{q\theta\theta} > 0$, it follows that $\int_{q(\hat{\theta})}^{q(\theta)} c_{q\theta}(y, x)\, dy < 0$ for any $x \in (\theta', \theta)$. Hence, $\Phi(\theta', \hat{\theta}) = \Phi(\theta, \hat{\theta}) + \int_{\theta'}^{\theta} \int_{q(\hat{\theta})}^{q(\theta)} c_{q\theta}(y, x)\, dy\, dx < 0$ and (NLIC) is violated. □

**Proof of theorem 1:** To show that $q \geq q^r$ suppose, by way of contradiction, that $q(\theta') < q^r(\theta')$ for some $\theta' \in \Theta$. The set $\{\theta : q(\theta) < q^r(\theta)\}$ must then have positive Lebesgue measure because $q$ is increasing and càdlàg and $q^r$ is continuous. Consider the decision function $q^*(\theta)$ where $q^*(\theta) = max\{q(\theta), q^r(\theta)\}$.

$q^*$ achieves a strictly higher objective value than $q$ in problem (SB) because $q^r(\theta)$ is defined as the pointwise maximizer of the objective. The improvement is strict as $q$ and $q^*$ differ on a set of positive measure and the objective is strictly concave in $q$ (see lemma 2). This will contradict the optimality of $q$ if $q^*$ also satisfies (NLIC).

Since $q$ and $q^r$ are both increasing, $q^*$ is also increasing. By lemma 3, $q^*$ is incentive compatible if

$$\Phi^*(\theta,\hat{\theta}) = -\int_{\hat{\theta}}^{\theta}\int_{q^*(\hat{\theta})}^{q^*(x)} c_{q\theta}(y,x)\,dy\,dx \geq 0$$

for arbitrary types $\theta$ and $\hat{\theta} < \theta$. If $q^*(\hat{\theta}) = q(\hat{\theta})$, $q^*(x) \geq q(x)$ implies $\Phi^*(\theta,\hat{\theta}) \geq \Phi(\theta,\hat{\theta}) \geq 0$: as $q(x) \geq s(x)$ for all $x \in (\hat{\theta},\theta)$, $\Phi^*(\theta,\hat{\theta}) - \Phi(\theta,\hat{\theta}) = -\int_{\hat{\theta}}^{\theta}\int_{q(x)}^{q^*(x)} c_{q\theta}(y,x)\,dy\,dx \geq 0$.

If $q^*(\hat{\theta}) > q(\hat{\theta})$ (and therefore $q^*(\hat{\theta}) = q^r(\hat{\theta})$), there are three possibilities: (i) there exists a type $\theta' \in (\hat{\theta},\theta)$ with $q(\theta') = q^*(\hat{\theta})$, (ii) all types $\theta' \in (\hat{\theta},\theta)$ have $q(\theta') < q^*(\hat{\theta})$ and (iii) there are types $\theta' \in (\hat{\theta},\theta)$ with $q(\theta') > q^*(\hat{\theta})$ but no type $\theta'$ with $q(\theta') = q^*(\hat{\theta})$, hence $q$ is discontinuous on $[\hat{\theta},\theta)$.

If (i), then $\Phi(\theta,\theta') \geq 0$ implies incentive compatibility as $\Phi^*(\theta,\hat{\theta}) - \Phi(\theta,\theta') = \int_{\hat{\theta}}^{\theta'}\int_{q(\theta')}^{q^*(x)} -c_{q\theta}(y,x)\,dy\,dx + \int_{\theta'}^{\theta}\int_{q(x)}^{q^*(x)} -c_{q\theta}(y,x)\,dy\,dx \geq 0$ where the first double integral is positive because of $s(x) < s(\theta') \leq q(\theta') = q^*(\hat{\theta}) \leq q^*(x)$ for all $x \in (\hat{\theta},\theta')$ and the second one is positive by $q^*(x) \geq q(x) \geq s(x)$ for all $x \in (\theta',\theta)$.

In case (ii), $q^*(\theta') \geq q^*(\hat{\theta})$ for all $\theta' \in (\hat{\theta},\theta)$ since $q^*$ is increasing. The inequalities $q^*(\theta') \geq q^*(\hat{\theta}) > q(\theta') \geq s(\theta')$ imply then incentive compatibility as the integrand in $\Phi^*$ is non-positive for all arguments.

In case (iii), define $\theta' = sup\{x \in [\hat{\theta},\theta) : \ q(x) < q^*(\hat{\theta})\}$. Note that $q$ is discontinuous at $\theta'$. Incentive compatibility between $\theta$ and $\theta'$ under $q$ implies $\int_{\theta'}^{\theta}\int_{q^-(\theta')}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \leq 0$ as well as $\int_{\theta'}^{\theta}\int_{q^+(\theta')}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \leq 0$. From $c_{qq\theta} < 0$ and $q^-(\theta') \leq q^*(\hat{\theta}) \leq q^+(\theta')$, it follows that $\int_{\theta'}^{\theta}\int_{q^*(\hat{\theta})}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \leq 0$. Incentive compatibility is satisfied because $q^*(x) \geq q^*(\hat{\theta})$ for all $x \in (\hat{\theta},\theta')$ implies $\Phi^*(\theta,\hat{\theta}) \geq -\int_{\theta'}^{\theta}\int_{q^*(\hat{\theta})}^{q(x)} c_{q\theta}(y,x)\,dy\,dx \geq 0$. Hence, $q \geq q^r$.

Next we consider the properties of $\eta$. Define $\eta$ by equation (4). As $q$ is monotonic, it is differentiable almost everywhere and continuous almost everywhere. These two properties as well as the càdlàg property are then also inherited by $\eta$. It is shown in the supplementary material that $q$ is bounded which is then also true for $\eta$. As the left hand side of (4) is decreasing in $q$ (lemma 2) and $c_{q\theta}(q(\theta),\theta) < 0$ by $q \geq q^r$, $\eta(\theta) \geq 0$ follows from $q \geq q^r$.

To turn to the first bullet point, let $q$ be strictly increasing on an interval $(\theta_1,\theta_2)$ and assume that $\Phi(\theta',\hat{\theta}') > 0$ for every $\hat{\theta}' \in (\theta_1,\theta_2)$ and every $\theta' \in \Theta \setminus \{\hat{\theta}'\}$. Suppose, contrary to the theorem, that there exist types $\theta'$ and $\theta''$ both in $(\theta_1,\theta_2)$ with $\theta'' > \theta'$ and $\eta(\theta'') > \eta(\theta')$. Then, by the right continuity of $\eta$, there exists an $\varepsilon > 0$ and an $\tilde{\eta}$ such that $\eta(\theta) > \tilde{\eta}$ for all $\theta \in [\theta'',\theta''+\varepsilon]$ and $\eta(\theta) < \tilde{\eta}$ for all $\theta \in [\theta',\theta'+\varepsilon]$ as well as $\theta''+\varepsilon < \theta_2$. Consider changing the decision to

$$q^c(\theta,\varepsilon') = \begin{cases} \varepsilon' q(\theta'+\varepsilon) + (1-\varepsilon')q(\theta) & \text{for } \theta \in [\theta',\theta'+\varepsilon] \\ \delta(\varepsilon')q(\theta'') + (1-\delta(\varepsilon'))q(\theta) & \text{for } \theta \in [\theta'',\theta''+\varepsilon) \\ q(\theta) & \text{else} \end{cases}$$

where $\delta(\varepsilon') : \mathbb{R}_+ \to \mathbb{R}_+$ is chosen such that

$$\int_{\theta'}^{\theta'+\varepsilon}\int_{q(x)}^{q^c(x,\varepsilon')} c_{q\theta}(y,x)\;dy\;dx = \int_{\theta''}^{\theta''+\varepsilon}\int_{q^c(x,\varepsilon')}^{q(x)} c_{q\theta}(y,x)\;dy\;dx. \tag{8}$$

For $\varepsilon' \geq 0$ small enough, this equation defines $\delta$ uniquely as $q(\theta) \geq q^r(\theta) > s(\theta)$ by lemma 2 which implies that both integrands are negative for all arguments. Therefore, the left hand side is zero for $\varepsilon' = 0$ and strictly decreasing in $\varepsilon'$ by the strict monotonicity of $q$ on $(\theta_1,\theta_2)$. The right hand side is – through $q^c$ – strictly decreasing in $\delta$ and zero for $\delta = 0$. Note that $\delta(\varepsilon')$ is, by its definition (8), differentiable and its derivative is

$$\delta'(\varepsilon') = \frac{\int_{\theta'}^{\theta'+\varepsilon}\left[c_{q\theta}(q^c(x,\varepsilon'),x)(q(\theta'+\varepsilon)-q(x))\right]\;dx}{\int_{\theta''}^{\theta''+\varepsilon}\left[c_{q\theta}(q^c(x,\varepsilon'),x)(q(x)-q(\theta''))\right]\;dx} > 0.$$

The change in the principal's payoff caused by the change in the decision function is

$$\Delta(\varepsilon') = \int_{\underline{\theta}}^{\bar{\theta}} \int_{q(x)}^{q^c(x,\varepsilon')} ((u_q(y,x) - c_q(y,x))f(x) + (1-F(x))c_{q\theta}(y,x))\, dy\, dx.$$

Note that the integrand in the last expression equals $\eta(x)c_{q\theta}(q(x),x)$ if $y = q(x)$ by the definition of $\eta$ in (4). Now consider the derivative of $\Delta$ at $\varepsilon' = 0$:

$$\begin{aligned}\Delta'(0) =& \int_{\theta'}^{\theta'+\varepsilon} (\eta(x)c_{q\theta}(q(x),x)(q(\theta'+\varepsilon) - q(x)))\, dx + \int_{\theta''}^{\theta''+\varepsilon} (\eta(x)c_{q\theta}(q(x),x)(q(\theta'') - q(x))\delta'(0))\, dx \\ >& \tilde{\eta}\left(\int_{\theta'}^{\theta'+\varepsilon} (c_{q\theta}(q(x),x)(q(\theta'+\varepsilon) - q(x)))\, dx + \delta'(0)\int_{\theta''}^{\theta''+\varepsilon} (c_{q\theta}(q(x),x)(q(\theta'') - q(x)))\, dx\right) \\ =& \tilde{\eta}\left(\int_{\theta'}^{\theta'+\varepsilon} (c_{q\theta}(q(x),x)(q(\theta'+\varepsilon) - q(x)))\, dx - \int_{\theta'}^{\theta'+\varepsilon} (c_{q\theta}(q(x),x)(q(\theta'+\varepsilon) - q(x)))\, dx\right) = 0\end{aligned}$$

where the inequality holds because $\tilde{\eta} > \eta(x)$ for $x \in [\theta', \theta'+\varepsilon]$ and $\tilde{\eta} < \eta(x)$ for $x \in [\theta'', \theta''+\varepsilon]$ and because the integrand of the first (second) integral is negative (positive) for all $x$. As $\Delta(0) = 0$, the principal's payoff under $q^c(\cdot,\varepsilon')$ is higher than his payoff under $q$ for $\varepsilon' > 0$ sufficiently small.

The changed decision is implementable: by lemma 3, upwards incentive constraints cannot be violated since the changed decision inherits monotonicity from the original decision. Incentive constraints from types $\theta \notin (\theta', \theta''+\varepsilon)$ to types $\hat{\theta} \notin (\theta', \theta''+\varepsilon)$ are – by the definition of $\delta(\varepsilon')$ – not affected by the change. Incentive constraints from types $\theta \in (\theta', \theta''+\varepsilon)$ to types $\hat{\theta} < \theta$ are relaxed as $\Phi^c(\theta,\hat{\theta}) = \Phi(\theta,\hat{\theta}) - \int_{\theta'}^{\theta}\int_{q(x)}^{q^c(x)} c_{q\theta}(y,x)\, dy\, dx > \Phi(\theta,\hat{\theta})$ where $\Phi^c$ is $\Phi$ under the decision $q^c$; see figure 5.

By assumption, $\Phi(\theta,\hat{\theta}) > 0$ for all $\hat{\theta} \in (\theta', \theta''+\varepsilon)$ and $\theta \neq \hat{\theta}$ under the original decision and this also holds true for the changed decision if $\varepsilon' > 0$ is sufficiently small: for $\varepsilon'$ small enough, $q^c > s$ which implies that there exists an $\varepsilon'' > 0$ such that $\Phi^c(\theta,\hat{\theta}) > 0$ for all $\theta \in \Theta$ and $\hat{\theta} \in (\theta - \varepsilon'', \theta)$. The assumption $\Phi(\theta,\hat{\theta}) > 0$ for all $\hat{\theta} \in (\theta', \theta''+\varepsilon)$ and the continuity of $\Phi$ in its first argument imply that for a given $\hat{\theta} \in [\theta', \theta''+\varepsilon]$ we have $G(\hat{\theta}) \equiv \min_{\theta\in[\hat{\theta}+\varepsilon'',\bar{\theta}]} \Phi(\theta,\hat{\theta}) > 0$. I claim that also $\inf_{\hat{\theta}\in[\theta',\theta''+\varepsilon]} G(\hat{\theta}) > 0$. This is true by assumption if there exists a $\hat{\theta}$ minimizing $G$. Such a minimizer might not exist if $q$ is discontinuous. But, by the same arguments as in lemma 4, $\inf_{\hat{\theta}\in[\theta',\theta''+\varepsilon]} G(\hat{\theta}) = 0$ while there is no minimizer would contradict the incentive compatibility of $q$.

As $\Phi^c$ is continuous in $\varepsilon'$ and $\Phi^c = \Phi$ for $\varepsilon' = 0$, (NLIC) is slack for $\hat{\theta} \in [\theta', \theta''+\varepsilon]$ and $\theta > \hat{\theta}$ for $\varepsilon' > 0$ small enough. Therefore, $q^c$ is implementable and increases the principal's payoff for $\varepsilon' > 0$ small enough which contradicts the optimality of $q$. Hence, $\eta$ has to be decreasing on $(\theta_1, \theta_2)$.

The second statement is proven similarly. Let $q$ be strictly increasing on $(\theta_1, \theta_2)$ and let $\Phi(\theta', \hat{\theta}') > 0$ for every $\theta' \in (\theta_1, \theta_2)$ and every $\hat{\theta}' \in \Theta \setminus \{\theta'\}$. Suppose, contrary to the theorem, that there were $\theta_1 < \theta' < \theta'' < \theta_2$ such that $\eta(\theta') > \eta(\theta'')$. Then there exists an $\varepsilon > 0$ and $\tilde{\eta}$ such that $\eta(\theta) > \tilde{\eta}$ for $\theta \in [\theta', \theta'+\varepsilon]$ and $\eta(\theta) < \tilde{\eta}$ for $\theta \in [\theta'', \theta''+\varepsilon]$. Consider the changed decision

$$q^c(\theta,\varepsilon') = \begin{cases} \varepsilon' q(\theta') + (1-\varepsilon')q(\theta) & \text{for } \theta \in [\theta', \theta'+\varepsilon) \\ \delta(\varepsilon')q(\theta''+\varepsilon) + (1-\delta(\varepsilon'))q(\theta) & \text{for } \theta \in [\theta'', \theta''+\varepsilon) \\ q(\theta) & \text{else} \end{cases}$$

where $\delta(\varepsilon') : \mathbb{R}_+ \to \mathbb{R}_+$ is chosen such that (8) holds. This yields

$$\delta'(\varepsilon') = \frac{\int_{\theta'}^{\theta'+\varepsilon} [c_{q\theta}(q^c(x,\varepsilon'),x)(q(\theta') - q(x))]\, dx}{\int_{\theta''}^{\theta''+\varepsilon} [c_{q\theta}(q^c(x,\varepsilon'),x)(q(x) - q(\theta''+\varepsilon))]\, dx} > 0.$$

Defining $\Delta(\varepsilon')$ as above, its derivative at $\varepsilon'=0$ is then

$$\begin{aligned}\Delta'(0) =& \int_{\theta'}^{\theta'+\varepsilon} (\eta(x)c_{q\theta}(q(x),x)(q(\theta')-q(x)))\ dx + \int_{\theta''}^{\theta''+\varepsilon} (\eta(x)c_{q\theta}(q(x),x)(q(\theta''+\varepsilon)-q(x))\delta'(0))\ dx\\ >&\tilde{\eta}\left(\int_{\theta'}^{\theta'+\varepsilon} (c_{q\theta}(q(x),x)(q(\theta')-q(x)))\ dx + \delta'(0)\int_{\theta''}^{\theta''+\varepsilon} (c_{q\theta}(q(x),x)(q(\theta''+\varepsilon)-q(x)))\ dx\right)\\ =&0.\end{aligned}$$

The changed decision satisfies (NLIC): incentive constraints from types in $(\theta_1,\theta_2)$ cannot be violated by assumption for $\varepsilon'>0$ small enough. Note that for every $\hat{\theta}\in[\theta',\theta''+\varepsilon]$ there exists a type $\tilde{\theta}(\hat{\theta},\varepsilon')\in[\theta',\theta''+\varepsilon]$ such that either $q^c(\hat{\theta},\varepsilon')=q(\tilde{\theta}(\hat{\theta}))$ or $q^-(\tilde{\theta}(\hat{\theta}))\le q^c(\hat{\theta},\varepsilon')<q^+(\tilde{\theta}(\hat{\theta}))$. In the latter case, lemma 4 implies that $\Phi(\theta,\tilde{\theta}(\hat{\theta},\varepsilon'))>0$ and $\int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta}\int_{q^-(\tilde{\theta}(\hat{\theta},\varepsilon'))}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx>0$ for all $\theta>\tilde{\theta}(\hat{\theta},\varepsilon')$. By $c_{qq\theta}<0$, these two inequalities imply $\int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta}\int_{z}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx>0$ for all $z\in[q^-(\tilde{\theta}(\hat{\theta},\varepsilon')),q^+(\tilde{\theta}(\hat{\theta},\varepsilon'))]$.

Incentive constraints from $\theta\ge\theta''+\varepsilon$ to $\hat{\theta}\in[\theta'+\varepsilon,\theta''+\varepsilon]$ are not violated because $\Phi^c(\theta,\hat{\theta}) = \int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta}\int_{q^c(\hat{\theta},\varepsilon')}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx - \int_{\hat{\theta}}^{\tilde{\theta}(\hat{\theta},\varepsilon')}\int_{q^c(\hat{\theta},\varepsilon')}^{q^c(x,\varepsilon')} c_{q\theta}(y,x)\,dy\,dx - \int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta''+\varepsilon}\int_{q(x)}^{q^c(x,\varepsilon')} c_{q\theta}(y,x)\,dy\,dx \ge \int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta}\int_{q^c(\hat{\theta},\varepsilon')}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx\ge 0$ where the last inequality follows from incentive compatibility of $q$ and the previous paragraph. Incentive constraints from $\theta\ge\theta''+\varepsilon$ to $\hat{\theta}\in[\theta',\theta'+\varepsilon)$ are not violated because $\Phi^c(\theta,\hat{\theta}) = \int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta}\int_{q^c(\hat{\theta},\varepsilon')}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx + \int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta'+\varepsilon}\int_{max(q^c(\hat{\theta},\varepsilon'),q^c(x,\varepsilon'))}^{q(x)} c_{q\theta}(y,x)\,dy\,dx - \int_{\theta''}^{\theta''+\varepsilon}\int_{q(x)}^{q^c(x,\varepsilon')} c_{q\theta}(y,x)\,dy\,dx \ge \int_{\tilde{\theta}(\hat{\theta},\varepsilon')}^{\theta}\int_{q^c(\hat{\theta},\varepsilon')}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx\ge 0$ where the first inequality follows from the definition of $\delta$ and the second by incentive compatibility of $q$. Note that $\theta'$ and $\theta''$ can be chosen such that $c_{q\theta}(q(\theta'),\theta''+\varepsilon)\le 0$ which implies that $\Phi(\theta,\hat{\theta})>0$ for $\theta\in[\theta',\theta''+\varepsilon]$ and $\hat{\theta}\in[\theta',\theta)$. As $q^c$ is monotone, upward incentive constraints are slack (lemma 3) and for all other types $\Phi$ did not change when changing $q$ to $q^c$.

Turning to the next statement in the theorem, let $\Phi(\theta',\hat{\theta}')=0$ and let $q$ be strictly increasing in neighborhoods of $\theta'$ and $\hat{\theta}'$. Note that lemma 5 implies that there has to be an open neighborhood of $\theta'$ such that $\Phi(\theta,\hat{\theta})>0$ for all $\hat{\theta}$ in this neighborhood and $\theta\neq\hat{\theta}$: otherwise, there would be a sequence $((\theta_n,\hat{\theta}_n))_{n=1}^{\infty}$ such that $\Phi(\theta_n,\hat{\theta}_n)=0$ and $\lim_{n\to\infty}\hat{\theta}_n=\theta'$. But then – by the continuity[20] of $\Phi$ – $\Phi(\tilde{\theta},\theta')=0$ violating lemma 5 where $\tilde{\theta}$ is the limit of a converging subsequence of $(\theta_n)_{n=1}^{\infty}$. Such a convergent subsequence exists by the Bolzano-Weierstrass theorem. From the first result of theorem 1, it follows that $\eta$ is decreasing in an open neighborhood of $\theta'$ such that (NLIC) does not hold with equality to types in this neighborhood. Similarly, by lemma 5, there has to be an open neighborhood of $\hat{\theta}'$ such that (NLIC) holds with inequality from types in this neighborhood. The second result of theorem 1 implies that $\eta$ is increasing in this neighborhood.

Now turn to $\eta^-(\bar{\theta})=0$ (and therefore $q^-(\bar{\theta})=q^{fb}(\bar{\theta})$) whenever $\Phi(\bar{\theta},\hat{\theta})>0$ for all $\hat{\theta}\in\Theta\setminus\{\bar{\theta}\}$. Note that in this case $\Phi(\theta,\hat{\theta})>0$ for all $\theta\in(\bar{\theta}-\varepsilon,\bar{\theta}]$ and $\hat{\theta}\in\Theta\setminus\{\theta\}$ for some $\varepsilon>0$: otherwise, there would be a sequence $((\theta_n,\hat{\theta}_n))_{n=1}^{\infty}$ such that $\Phi(\theta_n,\hat{\theta}_n)=0$ and $\lim_{n\to\infty}\theta_n=\bar{\theta}$. But then $\Phi(\bar{\theta},\tilde{\theta})=0$ where $\tilde{\theta}<\bar{\theta}$ is the limit of a convergent subsequence of $(\hat{\theta}_n)_{n=1}^{\infty}$. (Note that $q$ – and therefore $\Phi$ – is continuous at $\tilde{\theta}$ as, by the argument of lemma 4, incentive compatibility of $q$ would be contradicted otherwise.)

Clearly, $q(\theta)$ of $\theta\in(\bar{\theta}-\varepsilon,\bar{\theta}]$ does not affect non-local incentive constraints of other types, see figure 1b for an illustration. Consequently, the principal's payoff would be maximized by setting $q(\theta)=q^r(\theta)$ and therefore $\eta(\theta)=0$ for $\theta\in(\bar{\theta}-\varepsilon,\bar{\theta}]$. Hence, the optimality of $q$ is contradicted if it can be shown that the monotonicity constraint is not binding on $(\bar{\theta}-\varepsilon,\bar{\theta}]$. Suppose to the contrary that $q(\theta)=q^b>q^{fb}(\bar{\theta})$ holds for types $\theta\in[\theta',\bar{\theta})$ for some $\theta'<\bar{\theta}$.[21] By lemma 4, $\Phi(\theta,\hat{\theta})>0$ for $\theta\in(\theta',\bar{\theta}]$ and $\hat{\theta}\notin[\theta',\bar{\theta}]$. First, note

[20] By lemma 4, $q$ can neither be discontinuous at $\hat{\theta}'$ nor at the limit of (a subsequence of) $(\hat{\theta}_n)_{n=1}^{\infty}$. Hence, $\Phi$ is continuous at these points.

[21] If types arbitrarily close to $\bar{\theta}$ are bunched on $q^b\le q^{fb}(\bar{\theta})$, clearly $q^-(\bar{\theta})\le q^{fb}(\bar{\theta})$ which – together with $\eta\ge 0$ – implies $\eta^-(\bar{\theta})=0$.

that $q$ has to be continuous at $\theta'$ as otherwise the principal's payoff could be increased by reducing $q^b$: as $q^b > q^{fb}(\bar{\theta}) \geq q^r(\theta)$ and the principal's objective in problem (SB) is concave in $q$ with maximum at $q^r$, lowering $q^b$ increases this objective. Given continuity of $q$ at $\theta'$, $\Phi(\theta,\hat{\theta}) > 0$ for $\theta \in [\theta'-\varepsilon', \theta']$ and $\hat{\theta} < \theta$ for some small $\varepsilon' > 0$ by the same argument as in the proof of lemma 4.[22] Given that $q(\theta) > q^{fb}(\bar{\theta}) > q^r(\theta)$ for all $\theta \in [\theta'-\varepsilon', \bar{\theta})$, the principal's payoff could be increased by changing $q(\theta)$ to $q(\theta'-\varepsilon')$ for all $\theta \in [\theta'-\varepsilon', \bar{\theta}]$ (assuming $\varepsilon' > 0$ is small enough). This contradicts the optimality of $q$. Hence, the monotonicity constraint cannot bind and the result follows.

The part that $\eta^+(\underline{\theta}) = 0$ if $\Phi(\theta,\underline{\theta}) > 0$ for all $\theta > \underline{\theta}$ is similar: for the same argument as above, there exists an $\varepsilon > 0$ such that $\Phi(\theta,\hat{\theta}) > 0$ for any $\hat{\theta} \in [\underline{\theta}, \underline{\theta}+\varepsilon)$ and $\theta \in \Theta \setminus \{\hat{\theta}\}$. Reducing $q(\theta)$ to $\varepsilon' q^r(\theta) + (1-\varepsilon')q(\theta)$ for all $\theta \in [\underline{\theta}, \underline{\theta}+\varepsilon/2)$ increases the principal's payoff and cannot violate the monotonicity constraint as $q(\theta) \geq q^r(\theta)$ and $q^r_\theta > 0$ by lemma 2. For $\varepsilon' > 0$ small enough, (NLIC) cannot be violated by the change.

Next, let $q$ be constant on $(\theta_1, \theta_2)$, i.e. $q(\theta) = q^b$ for all $\theta \in (\theta_1,\theta_2)$. Start with the simple case in which $q$ is continuous at $\theta_1$ and $\theta_2$ and $\theta_2 < \bar{\theta}$. Suppose contrary to the theorem that $\eta(\theta_1) > \eta(\theta_2)$. By lemma 4, (NLIC) cannot hold with equality from types in $(\theta_1,\theta_2)$ to types $\hat{\theta} \notin (\theta_1,\theta_2)$. Furthermore, there has to be an $\varepsilon > 0$ such that $\Phi(\theta,\hat{\theta}) > 0$ for $\hat{\theta} \neq \theta$ and $\theta \in (\theta_1-\varepsilon, \theta_1) \cup (\theta_2, \theta_2+\varepsilon)$: otherwise, $\Phi((\theta_1+\theta_2/2, \hat{\theta}) < 0$ by (C1) and $c_{q\theta\theta} > 0$.

Note that $\eta$ is continuous at $\theta_1$ and $\theta_2$ (as $q$ is) and therefore there exists $\tilde{\eta}$ such that $\eta(\theta) > \tilde{\eta}$ for $\theta \in [\theta'-\varepsilon, \theta'+\varepsilon]$ and $\eta(\theta) < \tilde{\eta}$ for $\theta \in [\theta_2-\varepsilon, \theta_2+\varepsilon]$ for $\varepsilon > 0$ small enough. Consider the following changed decision

$$q^c(\theta,\varepsilon') = \begin{cases} \varepsilon' q(\theta_1-\varepsilon) + (1-\varepsilon')q(\theta) & \text{for } \theta \in [\theta_1-\varepsilon, \theta_1+\varepsilon) \\ \delta(\varepsilon')q(\theta_2+\varepsilon) + (1-\delta(\varepsilon'))q(\theta) & \text{for } \theta \in [\theta_2-\varepsilon, \theta_2+\varepsilon] \\ q(\theta) & \text{else.} \end{cases}$$

where $\delta(\varepsilon')$ is chosen such that $\int_{\theta_1-\varepsilon}^{\theta_1+\varepsilon} \int_{q^c(\theta,\varepsilon')}^{q(\theta)} c_{q\theta}(y,x)\,dy\,dx = \int_{\theta_2-\varepsilon}^{\theta_2+\varepsilon} \int_{q(\theta)}^{q^c(\theta,\varepsilon')} c_{q\theta}(y,x)\,dy\,dx$. Following the same steps as in the proof of the second statement of this theorem, it is straightforward to show that $q^c$ increases the principal's payoff and is implementable. This contradicts the optimality of $q$.

If $q$ is discontinuous at $\theta_1$, let $q^c$ be $q^c(\theta) = q^b - \varepsilon'$ for $\theta \in [\theta_1, \theta_1+\varepsilon)$ and $q^c$ as in the last paragraph for $\theta \in [\theta_2-\varepsilon, \theta_2+\varepsilon]$ (and $q(\theta)$ for all other types). The same argument as above applies for $\varepsilon > 0$ and $\varepsilon' > 0$ small enough and $\delta$ chosen such that $\int_{\theta_1}^{\theta_1+\varepsilon} \int_{q^c(\theta,\varepsilon')}^{q(\theta)} c_{q\theta}(y,x)\,dy\,dx = \int_{\theta_2-\varepsilon}^{\theta_2+\varepsilon} \int_{q(\theta)}^{q^c(\theta,\varepsilon')} c_{q\theta}(y,x)\,dy\,dx$. If $q$ is discontinuous at $\theta_2$ or if $\theta_2 = \bar{\theta}$, the changed decision around $\theta_2$ has to be $q^c(\theta) = q^b + \varepsilon'$ for $\theta \in [\theta_2-\varepsilon, \theta_2]$ (while $q^c$ around $\theta_1$ as before). In both cases, this change increases the payoff of the principal if $\eta^+(\theta_1) > \eta^-(\theta_2)$. Since this contradicts optimality, $\eta^+(\theta_1) \leq \eta^-(\theta_2)$ as had to be shown.

Last, let $q$ be discontinuous at $\theta'$, i.e. $q^-(\theta') < q^+(\theta')$. If $q$ is such that there is no type $\tilde{\theta} \neq \theta'$ such that either $q^+(\theta') = q(\tilde{\theta})$ or $q^-(\theta') = q(\tilde{\theta})$, then lemma 4 implies that $\Phi(\theta,\theta') > 0$ for all $\theta \neq \theta'$. The same argument as in lemma 4 implies that there exists an open neighborhood of $\theta'$ such that $\Phi(\theta,\hat{\theta}) > 0$ for all $\hat{\theta} \neq \theta$. If $q$ is strictly increasing in an open neighborhood of $\theta'$, the first part of theorem 1 implies that $\eta$ is decreasing and therefore $\eta^-(\theta') \geq \eta^+(\theta')$.

Hence, only the following cases remain for this proof: (i) $q$ is constant on an interval $(\theta_1,\theta')$, (ii) $q$ is constant on an interval $(\theta',\theta_2)$, (iii) both (i) and (ii) are true and (iv) there exist intervals of types arbitrarily close to $\theta'$ on which q is constant, but $\theta'$ itself does not lie in any such interval, nor is it an endpoint of such an interval. Define $q^a = (q^-(\theta') + q^+(\theta'))/2$. Suppose, contrary to the theorem, that $\eta^-(\theta') < \eta^+(\theta')$. As both one sided limits of $\eta$ at $\theta'$ exist, there exists an $\varepsilon > 0$ such that $\eta(\theta) < (\eta^-(\theta') + \eta^+(\theta'))/2$ for all $\theta \in (\theta'-\varepsilon, \theta')$ and $\eta(\theta) > (\eta^-(\theta') + \eta^+(\theta')/2$ for $\theta \in (\theta', \theta'+\varepsilon)$. Consider now the following changed decision

$$q^c(\theta,\varepsilon') = \begin{cases} \varepsilon' q^a + (1-\varepsilon')q(\theta) & \text{for } \theta \in [\theta'-\varepsilon, \theta') \\ \delta(\varepsilon')q^a + (1-\delta(\varepsilon'))q(\theta) & \text{for } \theta \in [\theta', \theta'+\varepsilon) \\ q(\theta) & \text{else} \end{cases}$$

[22]In a nutshell, (C1) and $c_{q\theta\theta} > 0$ would imply that $\Phi(\bar{\theta},\hat{\theta}) < 0$ for some $\hat{\theta}$ if (NLIC) was holding with equality from types arbitrarily close to $\theta'$.

where $\delta(\varepsilon')$ is chosen such that (8) holds. Following the same steps as above, it is straightforward that there exist $\varepsilon > 0$ and $\varepsilon' > 0$ such that the principal's payoff is higher under $q^c$ than under $q$.

It remains to show that $q^c$ is incentive compatible in each of the three cases. Note that only incentive constraints to types in $[\theta' - \varepsilon, \theta' + \varepsilon)$ have to be checked as all other incentive constraints are not affected or even relaxed. If there exists an open neighborhood of $\theta'$ such that for all $\hat{\theta}$ in this neighborhood $\Phi(\theta, \hat{\theta}) > 0$ for all $\theta$ such that $q(\theta) \neq q(\hat{\theta})$, then (NLIC) will be satisfied in the changed decision for $\varepsilon' > 0$ small enough: as $q^c > s$, there exists an $\varepsilon'' > 0$ such that $\Phi(\theta, \hat{\theta}) > 0$ for all types such that $0 < \theta - \hat{\theta}^m(\hat{\theta}) < \varepsilon''$ where $\hat{\theta}^m(\hat{\theta}) = max\{\theta :\ q^-(\theta) = q(\hat{\theta})\}$. If a neighborhood of $\theta'$ as described exists, then $\min_{\theta \in [\hat{\theta}^m(\hat{\theta}) + \varepsilon'', \bar{\theta}],\, \hat{\theta} \in [\theta' - \varepsilon, \theta' + \varepsilon]} \Phi(\theta, \hat{\theta}) > 0$ for $\varepsilon > 0$ small enough. This implies that the changed decision is implementable for $\varepsilon' > 0$ small enough because $\Phi^c = \Phi$ for $\varepsilon' = 0$ and $\Phi^c$ is continuous in $\varepsilon'$. Hence, it remains to check whether $\Phi(\theta, \hat{\theta}) = 0$ for $\hat{\theta}$ arbitrarily close to $\theta'$ and $\theta$ such that $q(\theta) \neq q(\hat{\theta})$ and whether the changed decision is implementable if this is the case.

In case (iv), it was already established (3 paragraphs above) that there is an open neighborhood of $\theta'$ such that $\Phi(\theta, \hat{\theta}) > 0$ for all $\hat{\theta}$ in this neighborhood and $\theta$ such that $q(\theta) \neq q(\hat{\theta})$.

In case (i), (NLIC) cannot hold with equality for $\hat{\theta} = \theta' < \theta$: if $\Phi(\theta, \theta') = 0$, then $\int_{\theta'}^{\theta} c_{q\theta}(q(\theta'), x)\, dx \geq 0$ as otherwise $\Phi(\theta, \theta' + \varepsilon) < 0$. But then $\Phi(\theta, \theta' - \varepsilon) = \Phi(\theta, \theta') - \int_{\theta'}^{\theta} \int_{q^-(\theta')}^{q^+(\theta')} c_{q\theta}(y, x)\, dy\, dx < 0$ by $c_{qq\theta} < 0$ which contradicts incentive compatibility. For the same reason, (NLIC) will not hold with equality to types $\hat{\theta} \in (\theta', \theta' + \varepsilon)$ for $\varepsilon > 0$ small. Hence, in case (i), (NLIC) has to be checked only for $\theta > \theta'$ and $\hat{\theta} \in [\theta' - \varepsilon, \theta')$.

Let $\varepsilon$ be small enough such that $q(\theta' - \varepsilon) = q^-(\theta')$ and let $\Phi(\theta, \theta' - \varepsilon) = 0$. Then $\int_{\theta'}^{\theta} c_{q\theta}(q^-(\theta'), x)\, dx > 0$: otherwise $\Phi(\theta, \theta') = \Phi(\theta, \theta' - \varepsilon) - \int_{\theta'}^{\theta} \int_{q^-(\theta')}^{q^+(\theta')} c_{q\theta}(y, x)\, dy\, dx < 0$ because $c_{qq\theta} < 0$ implies $\int_{\theta'}^{\theta} c_{q\theta}(q^-(\theta'), x)\, dx > \int_{\theta'}^{\theta} c_{q\theta}(y, x)\, dx$ for $y > q^-(\theta')$ and $\Phi(\theta, \theta' - \varepsilon)$ is 0 by assumption. Now let $\varepsilon > 0$ be small enough such that $\int_{\theta' - \varepsilon}^{\theta} c_{q\theta}(q^-(\theta'), x)\, dx > 0$. The changed contracts for types $\hat{\theta} \in [\theta' - \varepsilon, \theta']$ is then less attractive for $\theta$: $\Phi^c(\theta, \hat{\theta}) = \Phi^c(\theta, \theta' - \varepsilon) + \int_{\theta' - \varepsilon}^{\theta} \int_{q(\theta' - \varepsilon)}^{min\{q^c(x), q^c(\hat{\theta})\}} c_{q\theta}(y, x)\, dy\, dx = \Phi(\theta, \theta' - \varepsilon) + \int_{\theta' - \varepsilon}^{\theta} \int_{q(\theta' - \varepsilon)}^{min\{q^c(x), q^c(\hat{\theta})\}} c_{q\theta}(y, x)\, dy\, dx > 0$ for $\varepsilon' > 0$ small because $\int_{\theta' - \varepsilon}^{\theta} c_{q\theta}(q^-(\theta'), x)\, dx > 0$ implies $\int_{\theta' - \varepsilon}^{\theta} c_{q\theta}(y, x)\, dx > 0$ for $y \in (q(\theta' - \varepsilon), min\{q^c(x), q^c(\hat{\theta})\})$. Hence, the change relaxes (NLIC). Cases (ii) and (iii) are similar to (i) and therefore relegated to the webappendix. □

*Proofs of section 4*

Some preliminary results are used repeatedly in the following proofs and therefore stated separately here. They are shown under the assumption of section 4, i.e. the solution satisfies continuity, strict monotonicity and (UQ).

**Lemma 6.** *The following four sets are closed: the sets $\{\theta :\ \Phi(\theta, \hat{\theta}) = 0$ for some $\hat{\theta} \neq \theta\}$ and $\{\hat{\theta} :\ \Phi(\theta, \hat{\theta}) = 0$ for some $\theta \neq \hat{\theta}\}$, the set of types from which (NLIC) binds and the set to which (NLIC) binds.*

**Proof:** Let the sequence $(\theta_n)_{n=1}^{\infty}$ converge to $\theta'$. Assume that for each $\theta_n$ there exists a $\hat{\theta}_n$ such that $\Phi(\theta_n, \hat{\theta}_n) = 0$. By the Bolzano Weierstrass theorem, there exists a convergent subsequence of $(\hat{\theta}_n)_{n=1}^{\infty}$. Call its limit $\hat{\theta}'$. By the continuity of $\Phi$ and as $\Phi(\theta_n, \hat{\theta}_n) = 0$ for all $n$, $\Phi(\theta', \hat{\theta}') = 0$. If (NLIC) binds from each $\theta_n$, $\eta$ cannot be constant in any open neighborhood of any $\theta_n$. Hence, $\eta$ cannot be constant in any open neighborhood of $\theta'$ while being decreasing in a neighborhood of $\theta'$ by theorem 1. Hence, (NLIC) binds from $\theta'$. The proof for $\{\hat{\theta} :\ \Phi(\theta, \hat{\theta}) = 0$ for some $\theta \neq \hat{\theta}\}$ and the set of $\hat{\theta}$ to which (NLIC) binds is analogous. □

**Lemma 7.** *(NLIC) binds from $\theta'$ if for some $\varepsilon > 0$ either (i) $\eta(\theta) > \eta(\theta')$ for all $\theta \in (\theta' - \varepsilon, \theta')$ and $\eta(\theta) \leq \eta(\theta')$ for all $\theta \in (\theta', \theta' + \varepsilon)$ or (ii) $\eta(\theta) \geq \eta(\theta')$ for all $\theta \in (\theta' - \varepsilon, \theta')$ and $\eta(\theta) < \eta(\theta')$ for all $\theta \in (\theta', \theta' + \varepsilon)$.*
*(NLIC) binds to $\theta'$ if either (i) $\eta(\theta) < \eta(\theta')$ for all $\theta \in (\theta' - \varepsilon, \varepsilon')$ and $\eta(\theta) \geq \eta(\theta')$ for all $\theta \in (\theta', \theta' + \varepsilon)$ or (ii) $\eta(\theta) \leq \eta(\theta')$ for all $\theta \in (\theta' - \varepsilon, \varepsilon')$ and $\eta(\theta) > \eta(\theta')$ for all $\theta \in (\theta', \theta' + \varepsilon)$.*

**Proof:** Let the conditions of the first statement be satisfied. Then $\eta$ is not increasing in any neighborhood of $\theta'$. By the last statement in the third bullet of theorem 1, (NLIC) can then not bind to $\theta'$. Also $\eta$ cannot be constant in any open neighborhood of $\theta'$. Hence, (NLIC) cannot be slack at $\theta'$ as lemma 6 implies that the set of types at which (NLIC) is slack is open. Consequently, (NLIC) binds from $\theta'$. The second statement is proven similarly. □

**Lemma 8.** *If (NLIC) binds from $\bar{\theta}$, then $\eta(\bar{\theta}) = \eta(\hat{\theta}^m)$ where $\hat{\theta}^m = \max \hat{\Theta}^m$ and $\hat{\Theta}^m = \{\hat{\theta} :$ (NLIC) binds from $\bar{\theta}$ to $\hat{\theta}\}$. Furthermore, $\eta$ is increasing on $[\min \hat{\Theta}^m, \max \hat{\Theta}^m]$.*
*If (NLIC) binds to $\underline{\theta}$, $\eta(\underline{\theta}) = \eta(\theta^m)$ where $\theta^m = \min \Theta^m$ and $\Theta^m = \{\theta :$ (NLIC) binds from $\theta$ to $\underline{\theta}\}$ Furthermore, $\eta$ is decreasing on $[\min \Theta^m, \max \Theta^m]$.*

**Proof:** Take $\hat{\theta}_1, \hat{\theta}_2 \in [\min \hat{\Theta}^m, \max \hat{\Theta}^m]$ such that $\hat{\theta}_2 > \hat{\theta}_1$. By lemma 5, (NLIC) cannot bind from any type in $(\hat{\theta}_1, \hat{\theta}_2)$. Hence, theorem 1 implies that $\eta$ is increasing on $(\hat{\theta}_1, \hat{\theta}_2)$. By the continuity of $\eta$, $\eta(\hat{\theta}_1) \leq \eta(\hat{\theta}_2)$.

If (NLIC) is slack from and to all types in $(\hat{\theta}^m, \bar{\theta})$, then $\eta(\bar{\theta}) = \eta(\hat{\theta}^m)$ by theorem 1 as $\eta$ is constant on $(\hat{\theta}^m, \bar{\theta})$. If (NLIC) binds from a $\theta' \in (\hat{\theta}^m, \bar{\theta})$ to $\hat{\theta}'$ then $\hat{\theta}' \in (\hat{\theta}^m, \bar{\theta})$ by lemma 5 (and similarly if (NLIC) binds to $\hat{\theta}' \in (\hat{\theta}^m, \bar{\theta})$ to $\theta'$, then $\theta' \in (\hat{\theta}^m, \bar{\theta})$). For any such pair $(\theta', \hat{\theta}')$, (C3) implies $\eta(\theta') = \eta(\hat{\theta}')$. Let $\Theta'$ ($\hat{\Theta}'$) be the set of types in $(\hat{\theta}^m, \bar{\theta})$ from (to) which (NLIC) binds. By (C3), lemma 5 and the continuity of $\eta$, $\eta(\sup \Theta') = \eta(\inf \hat{\Theta}')$. By theorem 1, lemma 5 and the continuity of $\eta$, $\eta(\bar{\theta}) = \eta(\sup \Theta')$ and $\eta(\inf \hat{\Theta}') = \eta(\hat{\theta}^m)$. Therefore, $\eta(\bar{\theta}) = \eta(\hat{\theta}^m)$. The proof for $\underline{\theta}$ is analogous. □

**Lemma 9.** *Let (NLIC) bind from $\theta'$ to $\hat{\theta}'$. $\eta(\theta) \geq \eta' \equiv \min(\eta(\theta'), \eta(\hat{\theta}'))$ for all $\theta \in [\hat{\theta}', \theta']$.*

**Proof:** First, let $\eta' = \eta(\theta') = \eta(\hat{\theta}')$ (which is implied by (C3) if $\hat{\theta}'$ and $\theta'$ are interior). Suppose, to the contrary, there was a type $\tilde{\theta} \in [\hat{\theta}', \theta']$ with $\eta(\tilde{\theta}) < \eta'$. Denote $\tilde{\theta}' = sup\{\theta : \eta(\theta) \leq (\eta' + \eta(\tilde{\theta}))/2$ and $\theta \in [\hat{\theta}', \theta']\}$ and note that $\eta(\tilde{\theta}') = (\eta' + \eta(\tilde{\theta}))/2$ as $\eta$ is continuous (by the continuity of $q$). Then, (NLIC) binds to $\tilde{\theta}'$; see lemma 7. The type $\tilde{\theta}''$ from which (NLIC) binds to $\tilde{\theta}'$ has to be in $(\tilde{\theta}', \theta')$ by lemma 3 and 5 and therefore (C3) implies that $\eta(\tilde{\theta}'') = (\eta' + \eta(\tilde{\theta}))/2$ contradicting the definition of $\tilde{\theta}'$.

Second, let $\theta' = \bar{\theta}$. By lemma 8, $\eta(\bar{\theta}) = \eta(\hat{\theta}^m)$ where $\hat{\theta}^m$ is defined in lemma 8. The first part of the proof implies $\eta(\theta) \geq \eta(\bar{\theta})$ for all $\theta \in (\hat{\theta}^m, \bar{\theta})$. As lemma 8 establishes that $\eta$ is increasing on $[\min \hat{\Theta}^m, \max \hat{\Theta}^m]$, the result also holds for other $\hat{\theta}$ to which (NLIC) binds from $\bar{\theta}$. The proof for the case $\hat{\theta}' = \underline{\theta}$ is analogous. □

**Proof of theorem 2:** Take two types $\theta'$ and $\hat{\theta}'$ such that (NLIC) is binding from $\theta'$ to $\hat{\theta}'$ under the optimal decision $q$. By (C3), $\eta(\theta') = \eta(\hat{\theta}') \equiv \eta'$ if $\hat{\theta}'$ and $\theta'$ are interior.

As described in the text, $(\theta', \hat{\theta}')$ is a local minimizer of $\Phi^{\eta'}$: by (C1) and (C2), the first-order condition of minimizing $\Phi^\eta$ are satisfied. By lemmas 8 and the fact that $q(\cdot, \eta)$ is increasing in $\eta$, this also holds true if $\hat{\theta}'$ or $\theta'$ are boundary types: say $\theta' = \bar{\theta}$, then lemma 8 implies the statement for $\hat{\theta}' = \hat{\theta}^m$ (where $\hat{\theta}^m$ is defined in lemma 8). If $\hat{\theta}' \neq \hat{\theta}^m$, then lemma 5 and theorem 1 imply that $\eta(\hat{\theta}') \leq \eta(\hat{\theta}^m) = \eta(\bar{\theta})$. Consequently, $q(\bar{\theta}) \geq q(\bar{\theta}, \eta(\hat{\theta}'))$. This together with (C1) implies $\int_{q(\hat{\theta}')}^{q(\bar{\theta}, \eta(\hat{\theta}'))} c_{q\theta}(y, \bar{\theta})\, dy \geq 0$, i.e. the first-order condition for minimizing $\Phi^{\eta(\hat{\theta}')}$ is satisfied. The proof for $\underline{\theta}$ is analogous.

To verify that $(\theta', \hat{\theta}')$ is a minimizer, note that $q(\theta) \geq q(\theta, \eta')$ for $\theta \in (\hat{\theta}', \hat{\theta}' + \varepsilon) \cup (\theta' - \varepsilon, \theta')$ and $q(\theta) \leq q(\theta, \eta')$ for $\theta \in (\hat{\theta} - \varepsilon, \hat{\theta}) \cup (\theta', \theta' + \varepsilon)$ for $\varepsilon > 0$ small enough because $\eta$ is decreasing (increasing) in a neighborhood of $\theta'$ (of $\hat{\theta}'$); see theorem 1. Take $\theta'' \in (\theta', \theta' + \varepsilon)$ and $\hat{\theta}'' \in (\hat{\theta}', \hat{\theta}' + \varepsilon)$. As $q$ and $q(\cdot, \eta')$ are continuous, there exists a $\hat{\theta}''' \in (\hat{\theta}', \hat{\theta}'')$ such that $q(\hat{\theta}''') = q(\hat{\theta}'', \eta')$. Then,

$$\Phi^\eta(\theta'', \hat{\theta}'') - \Phi^\eta(\theta', \hat{\theta}') = \Phi(\theta'', \hat{\theta}''') - \Phi(\theta', \hat{\theta}') + \int_{\hat{\theta}'''}^{\hat{\theta}''} \int_{q(x,\eta')}^{q(\hat{\theta}''')} -c_{q\theta}(y, x)\, dy\, dx$$
$$+ \int_{\hat{\theta}'}^{\hat{\theta}'''} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx + \int_{\theta'}^{\theta''} \int_{q(x)}^{q(x,\eta')} -c_{q\theta}(y, x)\, dy\, dx \geq \Phi(\theta'', \hat{\theta}''') - \Phi(\theta', \hat{\theta}') \geq 0$$

where the first inequality follows from $q(x) > s(x)$ and $q(x, \eta') > s(x)$. The second inequality follows from $\Phi(\theta'', \hat{\theta}''') \geq 0 = \Phi(\theta', \hat{\theta}')$ by incentive compatibility. Similar inequalities hold for $\theta'' \in (\theta' - \varepsilon, \theta')$ or $\hat{\theta}'' \in (\theta' - \varepsilon, \hat{\theta}')$ or both; see the webappendix for details. Hence, $(\theta', \hat{\theta}')$ minimizes $\Phi^\eta$ locally.

Now, suppose that $(\theta', \hat{\theta}')$ does not minimize $\Phi^{\eta'}(\theta, \hat{\theta})$ on $[\hat{\theta}', \theta']$ and call the minimizer $(\theta'', \hat{\theta}'')$. Since $q$ is continuous and $\eta(\theta) \geq \eta'$ for all $\theta \in [\hat{\theta}', \theta']$ (lemma 9), there has to be a $\hat{\theta}''' \in [\hat{\theta}', \hat{\theta}'']$ such that $q(\hat{\theta}''') = q(\hat{\theta}'', \eta')$. Then,

$$\Phi(\theta', \hat{\theta}') = \Phi(\theta'', \hat{\theta}''') + \Phi^{\eta'}(\theta', \hat{\theta}') - \Phi^{\eta'}(\theta'', \hat{\theta}'') + \int_{\theta''}^{\theta'} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx + \int_{\hat{\theta}'}^{\hat{\theta}'''} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx$$
$$+ \int_{\hat{\theta}'''}^{\hat{\theta}''} \int_{q(x,\eta')}^{q(\hat{\theta}''')} -c_{q\theta}(y, x)\, dy\, dx \geq \Phi(\theta'', \hat{\theta}''') + \Phi^{\eta'}(\theta', \hat{\theta}') - \Phi^{\eta'}(\theta'', \hat{\theta}'') > 0$$

where the first inequality stems from $s(x) < q(x, \eta') \leq q(x)$ as $\eta(x) \geq \eta'$ for $x \in [\hat{\theta}', \theta']$. The second inequality is true by incentive compatibility between $\theta''$ and $\hat{\theta}'''$ and the assumption that $(\theta'', \hat{\theta}'')$ minimize $\Phi^{\eta'}$ on $[\hat{\theta}', \theta']$. As $\Phi(\theta', \hat{\theta}') = 0$ by assumption, the desired contradiction emerges. Hence, $(\theta', \hat{\theta}')$ minimize $\Phi^{\eta'}$ on $[\hat{\theta}', \theta']$. It was already shown above that $(\theta', \hat{\theta}')$ are local minimzers of $\Phi^{\eta'}$. Taking these two results together, the first claim in the proposition follows.

For the last claim, suppose contrary to the proposition that $\Phi^{\eta'}(\theta', \hat{\theta}') > 0$ and suppose (NLIC) was binding from $\theta'$ to $\hat{\theta}'$. Recall that $\eta' = min(\eta(\theta'), \eta(\hat{\theta}'))$ and the two arguments can differ if $\hat{\theta}' = \underline{\theta}$ or $\theta' = \bar{\theta}$. Then $\eta(\theta) \geq \eta'$ for all $\theta \in [\hat{\theta}', \theta']$ (see lemma 9). This implies $q(\theta) \geq q(\theta, \eta')$ for all $\theta \in [\hat{\theta}', \theta']$. Consequently, $\Phi(\theta', \hat{\theta}') \geq \Phi^{\eta'}(\theta', \hat{\theta}') > 0$, i.e. (NLIC) cannot bind from $\theta'$ to $\hat{\theta}'$. □

**Proof of proposition 1:** The proof starts with four preliminary results and proceeds then in three further steps.

Preliminary result 1: If – in the solution to problem (SB) – there exists a type $\theta_0$ such that $\eta(\theta_0) = \eta' > 0$, then there exists a type $\theta'$ with $\eta(\theta') = \eta'$ such that (NLIC) binds either to or from $\theta'$.

Let $\theta' = sup\{\theta : \eta(\theta) = \eta'\}$. As $\eta$ is continuous by the continuity of $q$, $\eta(\theta') = \eta'$.

Consider $\theta' < \bar{\theta}$: by the definition of $\theta'$ and the continuity of $\eta$, either (i) $\eta(\theta) < \eta'$ for all $\theta > \theta'$ or (ii) $\eta(\theta) > \eta'$ for all $\theta > \theta'$. In case (i), $\eta(\theta) \geq \eta'$ for $\theta \in (\theta' - \varepsilon, \theta')$ for some $\varepsilon > 0$: Suppose otherwise. Then $\eta$ is neither increasing nor decreasing in any open neighborhood of $\theta'$. By theorem 1, (NLIC) can therefore not bind from or to $\theta'$. Then, by lemma 6, there has to be an open neighborhood of $\theta'$ such that (NLIC) is slack for all types in this neighborhood. As $\eta$ is not constant in any open neighborhood of $\theta'$, this is impossible which is the desired contradiction. In case (ii), a similar argument applies. In both cases, the conditions of lemma 7 are satisfied.

If $\theta' = \bar{\theta}$, then by theorem 1 there exists a $\hat{\theta}'$ such that $\Phi(\theta', \hat{\theta}') = 0$ and (NLIC) binds from $\theta'$ to $\hat{\theta}'$.

Preliminary result 2: For $\eta \leq \bar{\eta}$, $\theta(\eta)$ is decreasing and $\hat{\theta}(\eta)$ is increasing in $\eta$. If $\theta(\eta) < \bar{\theta}$ (respectively $\hat{\theta}(\eta) > \underline{\theta}$) for $\eta \in (\eta'', \eta')$ with $0 \leq \eta'' < \eta' \leq \bar{\eta}$, the monotonicity of $\theta(\eta)$ (respectively $\hat{\theta}(\eta)$) is strict on $(\eta'', \eta')$.

Take $\eta' > \eta'' \geq 0$ with $\eta' \leq \bar{\eta}$ and let $\theta' = \theta(\eta')$ and $\hat{\theta}' = \hat{\theta}(\eta')$. Note that $\Phi^{\eta''}_{\theta}(\theta', \hat{\theta}') < 0$ and $\Phi^{\eta''}_{\hat{\theta}}(\theta', \hat{\theta}') > 0$ because (i) $(\theta', \hat{\theta}(\eta'))$ satisfy the first-order condition for minimizing $\Phi^{\eta'}$ and (ii) $q(\theta, \eta') > q(\theta, \eta'')$ for all types and $c_{qq\theta} < 0$. This implies that there has to be a local minimizer $(\theta^m, \hat{\theta}^m)$ of $\Phi^{\eta''}$ such that $\theta^m \geq \theta'$ and $\hat{\theta}^m \leq \hat{\theta}'$ (and strict inequalities if $\theta' < \bar{\theta}$, respectively $\hat{\theta}' > \underline{\theta}$): consider the auxiliary minimization problem $\Phi^{\eta''}$ on $[\theta', \tilde{\theta}] \times [\underline{\theta}, \hat{\theta}']$ where $\tilde{\theta} > \theta'$ is the highest type in the domain of the minimization of $\Phi^{\eta''}$ in step 2 of the algorithm, i.e. either $\tilde{\theta} = \bar{\theta}$ or – if the sufficient conditions of proposition 4 are satisfied – $\tilde{\theta}$ solves $1 - F(\tilde{\theta}) = \eta''$. The auxiliary minimization problem has a solution $(\theta^m, \hat{\theta}^m)$ as $\Phi^{\eta''}$ is continuous. As $\Phi^{\eta''}_{\theta}(\theta', \hat{\theta}) \leq \Phi^{\eta''}_{\theta}(\theta', \hat{\theta}') < 0$ for all $\hat{\theta} \leq \hat{\theta}'$, $\theta^m \neq \theta'$ unless $\theta' = \bar{\theta}$. Similarly, $\hat{\theta}^m \neq \hat{\theta}'$ unless $\hat{\theta}' = \underline{\theta}$ because $\Phi^{\eta''}_{\hat{\theta}}(\theta, \hat{\theta}') \geq \Phi^{\eta''}_{\hat{\theta}}(\theta', \hat{\theta}') > 0$ for all $\theta \geq \theta'$. This implies that $(\theta^m, \hat{\theta}^m)$ is also a local minimizer when $\Phi^{\eta''}$ is minimized over the full domain. By assumption, the only local minimizer in this problem is $(\theta(\eta''), \hat{\theta}(\eta''))$ which implies $\hat{\theta}^m = \hat{\theta}(\eta'') < \hat{\theta}(\eta') < \theta(\eta') < \theta(\eta'') = \theta^m$ and establishes the result.

Preliminary result 3: In step 2 of the algorithm, $\min_{\theta,\hat{\theta}} \Phi^{\eta}(\theta, \hat{\theta})$ is strictly increasing in $\eta$ for $\eta \leq \bar{\eta}$.

Let $\bar{\eta} \geq \eta' > \eta'' \geq 0$ and note from preliminary result 2 that $\hat{\theta}(\eta'') \leq \hat{\theta}(\eta') < \theta(\eta') \leq \theta(\eta'')$. For brevity, let $\theta' = \theta(\eta')$ and $\hat{\theta}' = \hat{\theta}(\eta')$. Then, $\Phi^{\eta'}(\theta', \hat{\theta}') = \Phi^{\eta''}(\theta', \hat{\theta}') + \int_{\hat{\theta}'}^{\theta'} \int_{q(\hat{\theta}', \eta'')}^{q(\hat{\theta}', \eta')} c_{q\theta}(y, x)\, dy\, dx -$

$\int_{\hat{\theta}'}^{\theta'} \int_{q(x,\eta'')}^{q(x,\eta')} c_{q\theta}(y,x)\,dy\,dx > \Phi^{\eta''}(\theta',\hat{\theta}') + \int_{\hat{\theta}'}^{\theta'} \int_{q(\hat{\theta}',\eta'')}^{q(\hat{\theta}',\eta')} c_{q\theta}(y,x)\,dy\,dx > \Phi^{\eta''}(\theta',\hat{\theta}') > \min_{\theta,\hat{\theta}} \Phi^{\eta''}(\theta,\hat{\theta})$ where the first inequality holds as $q(x,\eta') > q(x,\eta'') > s(\theta)$. The second inequality follows from $c_{qq\theta} < 0$ and the necessary first-order condition for minimizing $\Phi^{\eta'}$: $\int_{\hat{\theta}'}^{\theta'} c_{q\theta}(q(\hat{\theta}',\eta'),x)\,dx \geq 0$.

Preliminary result 4: $\bar{\eta} = \max \eta(\theta)$.

Let $\eta^{max} = \max \eta(\theta)$. Preliminary result 1 states that there have to be $\theta'$ and $\hat{\theta}'$ such that (NLIC) binds from $\theta'$ to $\hat{\theta}'$ and $\eta$ attains its maximum at one of the two types. In fact, $\eta(\theta') = \eta(\hat{\theta}') = \eta^{max}$: If both types are interior, this is implies by (C3). For boundary types, this is implied by lemma 8 and the definition of $\eta^{max}$. Hence, $\Phi^{\eta^{max}}(\theta',\hat{\theta}') \leq 0$ by theorem 2. This has to hold with equality as – by the definition of $\eta^{max}$ and the monotonicity of $q(\cdot,\eta)$ in $\eta$ – $\Phi^{\eta^{max}}(\theta',\hat{\theta}') \geq \Phi(\theta',\hat{\theta}') = 0$.

Step 1: The algorithm assigns the optimal decision to all types from which (NLIC) binds in the optimal mechanism.

If $\theta$ is an interior type from which (NLIC) binds, this follows immediately from theorem 2 and the assumption that $\Phi^{\eta}$ has a unique local minimizer. This leaves the case where (NLIC) is binding from $\bar{\theta}$. The algorithm assigns the highest $\eta \leq \bar{\eta}$ for which $\bar{\theta}$ is part of the $\Phi^{\eta}$ minimizer. Lemma 9 states that $\eta(\bar{\theta}) = \eta(\hat{\theta}^m)$ where $\hat{\theta}^m = max\{\hat{\theta} :$ (NLIC) binds from $\bar{\theta}$ to $\hat{\theta}\}$. Then, $(\bar{\theta},\hat{\theta}^m)$ are the unique local mininizer of $\Phi^{\eta(\bar{\theta})}$ by theorem 2. Also by theorem 2, $\Phi^{\eta(\bar{\theta})}(\bar{\theta},\hat{\theta}^m) \leq 0$ which implies $\eta(\bar{\theta}) \leq \bar{\eta}$ by preliminary result 3. Therefore, what remains to be shown is the following: if $\Phi^{\eta(\bar{\theta})}(\bar{\theta},\hat{\theta}^m) < 0$, then $\bar{\theta}$ is not part of a local minimizer of $\Phi^{\eta'}$ for any $\eta' \in (\eta(\bar{\theta}),\bar{\eta}]$.

By the continuity of $\eta$ and preliminary result 4, there is a type $\theta_0$ such that $\eta(\theta_0) = \eta'$. By preliminary result 1, there is a $\theta'$ with $\eta(\theta') = \eta'$ and (NLIC) binds to of from $\theta'$. If (NLIC) binds from $\theta''$ to $\theta'$, then $\theta'' \neq \bar{\theta}$ as $\eta(\bar{\theta}) < \eta(\theta')$ would contradict lemma 8. As $(\theta'',\theta')$ minimize $\Phi^{\eta'}$ by theorem 2 (and lemma 8 in case $\theta' = \underline{\theta}$), $\bar{\theta}$ will then not be part of the unique $\Phi^{\eta'}$ minimizer. If (NLIC) binds from $\theta'$ to $\hat{\theta}'$, then $\hat{\theta}' \neq \bar{\theta}$ as $\hat{\theta}' < \theta'$. Again by theorem 2 (and lemma 8 in case $\hat{\theta} = \underline{\theta}$), $(\theta',\hat{\theta}')$ minimize $\Phi^{\eta'}$. Hence, $\bar{\theta}$ is not part of the unique $\Phi^{\eta'}$ minimizer.

Step 2: The algorithm assigns the optimal decision to all types to which (NLIC) binds in the optimal mechanism. The proof is analogous to the proof of step 1.

Step 3: The algorithm assigns the optimal decision to all types where (NLIC) is slack in the optimal mechanism.

If a type $\theta'$ where (NLIC) is slack is not assigned a decision in step 4a, the algorithm assigns the optimal decision because theorem 1 ensures that $\eta$ is constant between $\theta'$ and the type closest to $\theta'$ from or to which (NLIC) binds. Hence, it is sufficient to show that $\theta'$ does not minimize $\Phi^{\eta'}$ for $\eta' \in [0,\bar{\eta}]$. Suppose otherwise for some $\eta'$. If there is a type $\theta_0$ such that $\eta(\theta_0) = \eta'$, then preliminary result 1 implies that there is a type $\theta''$ with $\eta(\theta'') = \eta'$ such that (NLIC) binds to or from $\theta''$. Given theorem 2 and the assumption that there is a unique local minimizer of $\Phi^{\eta'}$, this would contradict that (NLIC) is slack at $\theta'$. (By lemma 8, this is also true if $\theta''$ or $\theta_0$ are boundary types.)

If $\eta(\theta) > \eta'$ for all types, then (NLIC) binds from $\bar{\theta}$ to $\underline{\theta}$: by theorem 1, $\eta(\underline{\theta}) > 0$ implies that (NLIC) is binding to $\underline{\theta}$. Suppose, $\tilde{\theta} = sup\{\theta :$ (NLIC) binds from $\theta$ to $\underline{\theta}\} < \bar{\theta}$. By lemma 5 and the definition of $\tilde{\theta}$, there exists an $\varepsilon > 0$ such that (NLIC) is slack on $(\tilde{\theta},\tilde{\theta}+\varepsilon)$. Because of lemma 5, $\Phi(\theta,\hat{\theta}) > 0$ for any $\theta \geq \tilde{\theta}+\varepsilon$ and $\hat{\theta} \leq \tilde{\theta}$. Consequently, the decision of types in $(\tilde{\theta},\tilde{\theta}+\varepsilon)$ will not affect any binding non-local incentive constraint. Therefore, it is implementable to bunch all $\theta \in (\tilde{\theta},\tilde{\theta}+\varepsilon)$ on $q(\tilde{\theta})$. By $\eta(\theta) > \eta'$, we have $q(\theta) > q^r(\theta)$. Hence, the bunching increases the principal's payoff contradicting that $q$ is strictly increasing and optimal.

Hence, $(\bar{\theta},\underline{\theta})$ are local minimizers of $\Phi^{\eta''}$ for some $\eta'' > \eta'$; that is, $\int_{\underline{\theta}}^{\bar{\theta}} c_{q\theta}(q(\underline{\theta},\eta''),x)\,dx \geq 0$ and $\int_{q(\underline{\theta},\eta'')}^{q(\bar{\theta},\eta'')} c_{q\theta}(y,\bar{\theta})\,dy \geq 0$. As $q(\cdot,\eta)$ is increasing in $\eta$, this implies that $(\bar{\theta},\underline{\theta})$ is also a local minimizer of $\Phi^{\eta'}$. By assumption, the $\Phi^{\eta'}$ minimizer is unique. As (NLIC) is slack at $\theta'$, $\underline{\theta} \neq \theta' \neq \bar{\theta}$ which contradicts that $\theta'$ is part of the $\Phi^{\eta'}$ minimizer.

Finally, $\eta(\theta) < \eta'$ for all types is impossible: by the definition of $\bar{\eta}$ and preliminary result 3, incentive compatibility would be violated in this case. □

*Proofs of section 5*

**Proof of proposition 2:** Existence is shown in the webappendix. Here I show that any incentive compatible decision function $q$ which imposes decisions below $s$ for some type is dominated by the following changed decision

$$q^c(\theta) = \begin{cases} q(\theta) & \text{if } q(\theta) \geq s(\theta) \\ q^s(q(\theta), \theta) & \text{if } q(\theta) < s(\theta). \end{cases} \tag{9}$$

This changed decision is shown to be above $s$, increasing and incentive compatible.

First, it is shown that the principal's payoff is higher under $q^c$ than under $q$: the principal maximizes the expected value of $u(q, \theta) - c(q, \theta) + [(1 - F(\theta))/f(\theta)]c_\theta(q, \theta)$. If $q^s(q(\theta), \theta) \leq q^r(\theta)$, the principal's objective increases due to the change because of the concavity of (RP) and $q^r(\theta) > s(\theta)$ (recall that $q^r$ is the pointwise maximizer of the principal's objective). If $q^s(q(\theta), \theta) > q^r(\theta)$, then the same conclusion follows from $q^v(q(\theta), \theta) \geq q^s(q(\theta), \theta) > q^r(\theta)$ and the concavity of (RP).

Second, the changed decision $q^c$ is increasing: suppose otherwise, i.e. suppose there are $\theta' < \theta''$ with $q^c(\theta') > q^c(\theta'')$. Then for any $\varepsilon > 0$ there have to be $\theta'_\varepsilon$ and $\theta''_\varepsilon$ such that $\theta'_\varepsilon < \theta''_\varepsilon < \theta'_\varepsilon + \varepsilon$ and $q^c(\theta'_\varepsilon) > q^c(\theta''_\varepsilon)$. Let $t : \Theta \to \mathbb{R}$ denote the transfer function implementing $q$. Then incentive compatibility requires $c(q(\theta''_\varepsilon), \theta'_\varepsilon) - c(q(\theta'_\varepsilon), \theta'_\varepsilon) \geq t(\theta''_\varepsilon) - t(\theta'_\varepsilon) \geq c(q(\theta''_\varepsilon), \theta''_\varepsilon) - c(q(\theta'_\varepsilon), \theta''_\varepsilon)$. Combining the two inequalities yields

$$\int_{\theta'_\varepsilon}^{\theta''_\varepsilon} \int_{q(\theta''_\varepsilon)}^{q(\theta'_\varepsilon)} c_{q\theta}(y, x) \, dy \, dx \geq 0. \tag{10}$$

Denote by $\tilde{\Theta}$ the set $\{\theta : q(\theta) \geq s(\theta)\}$. Inequality (10) cannot be satisfied if $\theta'_\varepsilon < \theta''_\varepsilon$ and $q(\theta'_\varepsilon) > q(\theta''_\varepsilon) \geq s(\theta''_\varepsilon)$ as the integrand is then negative for all arguments. For $\varepsilon > 0$ small enough, inequality (10) is also violated if both $\theta'_\varepsilon$ and $\theta''_\varepsilon$ are in $\Theta \setminus \tilde{\Theta}$ as then $q(\theta'_\varepsilon) < q(\theta''_\varepsilon) < s(\theta''_\varepsilon)$ is implied by $q^c(\theta'_\varepsilon) > q^c(\theta''_\varepsilon)$ and $c_{q\theta\theta} > 0$. Therefore, one of the two types $\theta'_\varepsilon$ and $\theta''_\varepsilon$ has to be in $\tilde{\Theta}$ and one has to be in $\Theta \setminus \tilde{\Theta}$. Say $\theta'_\varepsilon \notin \tilde{\Theta}$ and $\theta''_\varepsilon \in \tilde{\Theta}$. Then $\int_{q(\theta'_\varepsilon)}^{q(\theta''_\varepsilon)} c_{q\theta}(y, \theta'_\varepsilon) \, dy = \int_{q^c(\theta''_\varepsilon)}^{q^c(\theta'_\varepsilon)} -c_{q\theta}(y, \theta'_\varepsilon) \, dy > 0$ as $q^c(\theta'_\varepsilon) > q^c(\theta''_\varepsilon) = q(\theta''_\varepsilon) \geq s(\theta''_\varepsilon) > s(\theta'_\varepsilon)$. By $c_{q\theta\theta} > 0$, this implies $\int_{q(\theta'_\varepsilon)}^{q(\theta''_\varepsilon)} c_{q\theta}(y, x) \, dy > 0$ for any $x \in [\theta'_\varepsilon, \theta''_\varepsilon]$. But then inequality (10) is violated. A similar argument can be made for $\theta'_\varepsilon \in \tilde{\Theta}$ and $\theta''_\varepsilon \notin \tilde{\Theta}$. Therefore, inequality (10) and thereby incentive compatibility is violated in any case contradicting $q^c(\theta'_\varepsilon) > q^c(\theta''_\varepsilon)$. Hence, $q^c$ is increasing.

Third, the changed decision $q^c$ is incentive compatible: since $q^c$ is increasing, only downward misrepresentation has to be considered (lemma 3). Note that the rent function $\pi$ is not affected by the change from $q$ to $q^c$ because of the definition of $q^s$ and lemma 1. Therefore, one has only to check whether any type wants to misrepresent as a lower type $\hat{\theta}$ at which $q(\hat{\theta}) < s(\hat{\theta})$. Since $\pi(\theta)$ is unchanged, (NLIC) under the changed decision can be written as

$$\begin{aligned}
\Phi^c(\theta, \hat{\theta}) &= \pi(\theta) - \pi(\hat{\theta}) - c(q^c(\hat{\theta}), \hat{\theta}) + c(q^c(\hat{\theta}), \theta) = \int_{\hat{\theta}}^{\theta} [-c_\theta(q(x), x) + c_\theta(q^c(\hat{\theta}), x)] \, dx \\
&= -\int_{\hat{\theta}}^{\theta} \int_{q^c(\hat{\theta})}^{q(x)} c_{q\theta}(y, x) \, dy \, dx = -\int_{\hat{\theta}}^{\theta} \int_{q^c(\hat{\theta})}^{q(\hat{\theta})} c_{q\theta}(y, x) \, dy \, dx - \int_{\hat{\theta}}^{\theta} \int_{q(\hat{\theta})}^{q(x)} c_{q\theta}(y, x) \, dy \, dx \\
&= \int_{\hat{\theta}}^{\theta} \int_{q(\hat{\theta})}^{q^c(\hat{\theta})} c_{q\theta}(y, x) \, dy \, dx + \Phi(\theta, \hat{\theta}) > 0
\end{aligned}$$

where the inequality follows from $\int_{q(\hat{\theta})}^{q^c(\hat{\theta})} c_{q\theta}(y, \hat{\theta}) \, dy = 0$ by the definition of $q^s$ and $c_{q\theta\theta} > 0$. □

**Proof of proposition 3:** Theorem 1 states that a solution could only be discontinuous at a type $\theta'$ where $\eta^-(\theta') \geq \eta^+(\theta')$. Given that $q$ is increasing, $q^-(\theta') < q^+(\theta')$ at a hypothetical discontinuity type $\theta'$. Using the definition of $\eta$ in (4), one can calculate the change in $\eta(\theta')$ at the discontinuity type

$$\begin{aligned}\eta^+(\theta')-\eta^-(\theta') &= \frac{[u_q(q^+(\theta'),\theta')-c_q(q^+(\theta'),\theta')]f(\theta')+(1-F(\theta'))c_{q\theta}(q^+(\theta'),\theta')}{c_{q\theta}(q^+(\theta'),\theta')}\\ &\quad-\frac{[u_q(q^-(\theta'),\theta')-c_q(q^-(\theta'),\theta')]f(\theta')+(1-F(\theta'))c_{q\theta}(q^-(\theta'),\theta')}{c_{q\theta}(q^-(\theta'),\theta')}\\ &=\int_{q^-(\theta')}^{q^+(\theta')}\frac{(u_{qq}-c_{qq})fc_{q\theta}+(1-F)c_{qq\theta}c_{q\theta}-(u_q-c_q)fc_{qq\theta}-(1-F)c_{q\theta}c_{qq\theta}}{c_{q\theta}^2}\,dx\end{aligned}$$

where the functions in the integrand are evaluated at $(x,\theta')$ and $\theta'$ respectively and the last equality uses the fundamental theorem of calculus. Note that the integrand is strictly positive for $x \le q^{fb}(\theta')$. If $x > q^{fb}(\theta')$, the integrand can be written as

$$\frac{f(u_q-c_q)}{c_{q\theta}}\left(\frac{u_{qq}-c_{qq}}{u_q-c_q}-\frac{c_{qq\theta}}{c_{q\theta}}\right)$$

which is also strictly positive due to the condition of the proposition. Hence, $\eta^-(\theta') < \eta^+(\theta')$ which contradicts theorem 1.

Using $q_1$ instead of $q^-(\theta')$, $q_2$ instead of $q^+(\theta')$ and $\eta^-(\theta')=\eta^+(\theta')=\eta'$ in the argument above shows that for a given type $\theta'$ and a given $\eta'$ there cannot be two $q$'s solving (4). Hence, (UQ) is satisfied.

Finally, strict monotonicity is shown. Suppose, contrary to the proposition, that there are $\theta_1$, $\theta_2$ such that $q(\theta)=q^b$ for all $\theta\in[\theta_1,\theta_2]$. By theorem 1 and the continuity of $q$, $\eta(\theta_1)\le\eta(\theta_2)$. But then $q(\theta_1)=q(\theta_1,\eta(\theta_1)) < q(\theta_2,\eta(\theta_1)) \le q(\theta_2,\eta(\theta_2)) = q(\theta_2)$ where the strict inequality follows from the assumption that $q(\cdot,\eta)$ is strictly increasing in type. This contradicts $q(\theta_1)=q(\theta_2)=q^b$. □

**Proof of proposition 4:** The optimal decision is continuous at types where it is below $q^{fb}$: the proof of proposition 3 shows that a discontinuity at a type $\theta$ where $q^+(\theta)\le q^{fb}(\theta)$ contradicts theorem 1.

Next it has to be shown that the decision is strictly monotone when it is below first best. This will be done in two steps. The first step is to show that $q_\theta(\theta)>0$ at types $\theta$ where $\eta$ is differentiable and $\eta_\theta(\theta)\ge 0$. The second step is to show that there are types $\theta$ at which $\eta_\theta(\theta)\ge 0$ in any hypothetical interval of types with the same decision. This yields a contradiction together with the first step.

First, the decision $q(\theta)$ has to satisfy the first order condition $[u_q-c_q]+c_{q\theta}(1-F-\eta)/f=0$ by theorem 1. For now, assume that $q$ (and therefore $\eta$) are continuously differentiable in a neighborhood of $\theta$. Then, given $\eta$, the sign of $q_\theta(\theta)$ can be determined from the implicit function theorem. Note that $q(\theta)\le q^{fb}(\theta)$ implies $1-F(\theta)-\eta(\theta)\ge 0$. Hence, the partial derivative of the left hand side of the first-order condition with respect to $q$ is negative. The sign of $q_\theta(\theta)$ is, therefore, the sign of the partial derivative of the left had side of the first-order condition with respect to $\theta$. Denoting $(1-F(\theta)-\eta(\theta))$ by $\lambda(\theta)$ this derivative is

$$u_{q\theta}(q(\theta),\theta)-c_{q\theta}(q(\theta),\theta)+\frac{\lambda(\theta)}{f(\theta)}c_{q\theta\theta}+\frac{\partial\ (\lambda(\theta)/f(\theta))}{\partial\theta}c_{q\theta}(q(\theta),\theta).$$

The first three terms are clearly positive as $q(\theta)\le q^{fb}(\theta)$ implies $\lambda(\theta)\ge 0$. The fourth term is positive if $\eta_\theta(\theta)\ge 0$ as then

$$\frac{\partial\ (\lambda(\theta)/f(\theta))}{\partial\theta}=\frac{-f^2(\theta)-f_\theta(\theta)(1-F(\theta))}{f^2(\theta)}-\frac{\eta_\theta(\theta)}{f(\theta)}+\frac{f_\theta(\theta)\eta(\theta)}{f^2(\theta)}\le 0$$

where the inequality stems from the monotone hazard rate assumption if $f_\theta(\theta)\le 0$. If $f_\theta(\theta)>0$, then $q^{fb}(\theta)\ge q(\theta)$ implies $\lambda(\theta)\ge 0$ which ensures the inequality above.

Now turn to the second step. Suppose contrary to the proposition that an interval $(\theta_1,\theta_2)$ exists such that $q(\theta)=q^b$ for all $\theta\in(\theta_1,\theta_2)$. This implies that $q$ – and therefore also $\eta$ – are differentiable on $(\theta_1,\theta_2)$. By theorem 1, $\eta^+(\theta_1)\le\eta^-(\theta_2)$. Consequently, there has to be some type $\theta\in(\theta_1,\theta_2)$ such that $\eta_\theta(\theta)\ge 0$. But then the first step shows that $q_\theta(\theta)>0$ which contradicts $q(\theta)=q^b$ for all $\theta\in(\theta_1,\theta_2)$.

The last part: $q$ is below $q^{fb}$ if $q^m$ increases and there is no distortion at the top. The proof is by contradiction. Suppose the optimal decision $q(\theta)$ was above the first best decision for some types. Since there is no distortion at the top by assumption and since $q$ is increasing, there has to be a type $\theta'$ such that $q(\theta') = q^{fb}(\theta')$ and $q(\theta) > q^{fb}(\theta)$ for $\theta \in (\theta' - \varepsilon, \theta')$ for some $\varepsilon > 0$.

Note that the arguments for continuity and strict monotonicity above are not tight at $q^{fb}$. Hence, for $\varepsilon > 0$ small enough, $q$ has to be continuous and strictly increasing on $(\theta' - \varepsilon, \theta')$. This implies that $\eta$ is also continuous on this interval. Recall that $q(\theta) > q^{fb}(\theta)$ if and only if $\eta(\theta) > 1 - F(\theta)$. Since $1 - F(\theta)$ is decreasing, it follows that $\eta(\theta') = 1 - F(\theta')$ and there does not exist an $\varepsilon' > 0$ such that $\eta$ is increasing on $(\theta' - \varepsilon', \theta')$.

Therefore, for each $n \in \mathbb{N}$ there has to be a type $\theta_n$ with $\theta' - 1/n < \theta_n < \theta'$ and $\Phi(\theta_n, \hat{\theta}_n) = 0$ for some $\hat{\theta}_n < \theta_n$ (theorem 1). By lemma 5, the sequence $(\hat{\theta}_n)_{n=1}^{\infty}$ is decreasing and therefore has a limit $\hat{\theta}'$. $\Phi(\theta', \hat{\theta}') = 0$ as $\Phi$ is continuous at $(\theta', \hat{\theta}')$ (following the argument of lemma 4, $\Phi$ cannot be discontinuous at $\hat{\theta}'$). As (C1) has to hold for each $(\theta_n, \hat{\theta}_n)$ and $q(\theta_n) > q^{fb}(\theta_n)$, $q(\hat{\theta}_n) < q^m(\theta_n)$ for each $n$. Hence, $q(\hat{\theta}_n) < q^m(\theta')$ as $q^m$ is increasing. In the limit, however, $q(\hat{\theta}') = q^m(\theta')$ has to hold by (C1) as $q(\theta') = q^{fb}(\theta')$.[23] Hence, $\hat{\theta}' < \hat{\theta}_n$ but $q(\hat{\theta}') > q(\hat{\theta}_n)$ which contradicts that $q$ is increasing. □


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# Webappendix

## Adverse selection without single crossing

Christoph Schottmüller

This webappendix contains supplementary material. First, there are several solved numerical examples. I give some details to the example of the main text and then turn to two other examples. The first example illustrates the use of the algorithm to numerically solve complicated problems where non-local incentive constraints bind from an interval of interior types. The second example features an optimal contract menu in which some types are bunched. Also the third derivative $c_{q\theta\theta}$ is negative in the relevant range leading to a downward sloping sign switching function $s$. As the third derivative does not change sign in the relevant range of $q$, the results from the paper still apply. The *Mathematica* (or *Python*) codes containing detailed calculations of all examples can be found online (`www.schottmueller.me`).

Second, the webappendix contains proofs that have been used in the same (or a very similar) way before. Existence of an optimal contract menu and optimality of non-stochastic contracts is shown similarly to Jullien [23]. The variational condition (C3) is shown as in Araujo and Moreira [33, 5].

### WA 1. Numerical Examples

*WA 1.1. Addendum to the example in the main text*

I skipped showing that $(\bar{\theta}, \underline{\theta})$ are the unique minimizers of all relevant $\Phi^\eta$. Showing that $(\bar{\theta}, \underline{\theta})$ satisfy (6) and (7) is easy: When plugging in $\theta = 1/4$ and $\hat{\theta} = 1/2$, (6) becomes

$$0 \leq \frac{9}{128} - \frac{6}{1280(2-\eta)} - \frac{126}{25600(2-\eta)^2}$$

which is true for $\eta \leq 1.7$ and therefore for all $\eta \leq \bar{\eta} = 1.67$. Similarly, (7) becomes

$$0 \leq \frac{1}{160}\left(10 - \frac{1}{2-\eta}\right)$$

which holds for all $\eta \leq 1.9$.

It remains to verify that $(\underline{\theta}, \bar{\theta})$ is the unique minimizer of $\Phi^\eta$ for all relevant $\eta$. First, I show that it is impossible that both (6) and (7) hold with equality, i.e. there is no pair of interior types minimizing $\Phi^\eta$. This is done by contradiction.

Suppose there are $1/4 < \hat{\theta}'' < \theta'' < 1/2$ that minimize $\Phi^\eta$ locally. As $q(\theta, \eta) > s(\theta)$ and since $\Phi^\eta(\theta, \theta) = 0$, there has to be a $(\hat{\theta}', \theta')$ with $\hat{\theta}'' < \hat{\theta}' < \theta' < \theta''$ such that $(\theta', \hat{\theta}')$ locally maximize $\Phi^\eta$: To see this, consider the maximization problem $\max_{\theta,\hat{\theta}} \Phi^\eta(\theta, \hat{\theta})$ over the domain $[\hat{\theta}'', \theta''] \times [\hat{\theta}'', \theta]$ (note the constraint $\hat{\theta} \leq \theta$). By the Weierstrass theorem, this problem has a solution $(\theta', \hat{\theta}')$. In this solution, $\hat{\theta}' < \theta'$ because $q(\theta, \eta) > s(\theta)$ and therefore $\Phi^\eta(\theta, \theta - \varepsilon) > \Phi^\eta(\theta, \theta) = 0$. Clearly, $(\theta', \hat{\theta}') \neq (\theta'', \hat{\theta}'')$ because $(\theta'', \hat{\theta}'')$ is a local minimizer of $\Phi^\eta$. Given that $\hat{\theta}' \neq \hat{\theta}''$, it is easy to see that $\theta' \neq \theta''$ because the strict monotonicity of $q(\cdot, \eta)$ implies that $\int_{q(\hat{\theta}')}^{q(\theta'')} c_{q\theta}(y, \theta'')\, dy \neq 0$ for any $\hat{\theta}' \neq \hat{\theta}''$, i.e. (7) cannot hold with equality for $\theta''$ and $\hat{\theta}' \neq \hat{\theta}''$. Similarly, $\hat{\theta}'' \neq \hat{\theta}'$ given that $\theta' \neq \theta''$ as (8) cannot hold for $\hat{\theta}''$ and $\theta' \neq \theta''$ with equality. Summing up, there exists a local $\Phi^\eta$ maximizer $(\theta', \hat{\theta}')$ such that $\hat{\theta}'' < \hat{\theta}' < \theta' < \theta''$.

As $(\hat{\theta}', \theta')$ is a local maximizer of $\Phi^\eta$, $(\hat{\theta}', \theta')$ also satisfy (6) and (7). Consider the function $g(\hat{\theta}) = \int_{\hat{\theta}}^{\theta''} c_{q\theta}(q(\hat{\theta}, \eta), x)\, dx$. Clearly, $g(\theta'') = 0$ and, by (7), $g(\hat{\theta}'') = 0$. As (7) is satisfied with equality for $(\hat{\theta}', \theta')$ and as $\theta' < \theta''$, we must have $g(\hat{\theta}') > 0$. Because $q(\theta'', \eta) > s(\theta'')$, $g(\theta'' - \varepsilon) < 0$ for $\varepsilon > 0$ small enough. Since $g$ is continuous, there has to exist some $\hat{\theta}''' \in (\hat{\theta}', \theta'')$ such that $g(\hat{\theta}''') = 0$. Consequently, $g$ is zero at

$\theta''$, $\hat{\theta}'''$ and $\hat{\theta}''$. I will now show that $g$ is convex for the relevant range of $\eta$ which implies that $g$ can attain each value at most twice. This will be the desired contradiction. Straightforward calculation gives

$$g''(\hat{\theta}) = \frac{1}{\theta''}\left(\frac{12\hat{\theta} - 4\theta''}{5(2-\eta)} + 2\right)$$

which is positive for $\eta < 2$.

Second, suppose that $\bar{\theta}$ is the solution for $\theta(\eta)$. Then it has to be shown that there is no interior minimizer $\hat{\theta}$ solving (7) with equality. Plugging in $\theta = 1/2$ into (7) gives after some rearranging

$$\frac{1}{2} - (1-\hat{\theta})2\hat{\theta} - (1-2\hat{\theta})\frac{2\hat{\theta}^2}{5(2-\eta)} = 0.$$

Note that the first-order condition is obviously satisfied for $\hat{\theta} = 1/2$ and that the left hand side of the last equation is strictly convex on $[1/4, 1/2]$ for $\eta < 2$, i.e. there is at most one interior solution. This solution will however be a $\Phi^\eta$ maximizer. This can be readily seen from the graph or the fact that $\Phi^\eta(1/2, 1/2-\varepsilon) > 0$ as $q^\eta > s$. Hence, there is no interior $\hat{\theta}(\eta)$.

Third, given that $\hat{\theta}(\eta) = \underline{\theta}$ it has to be shown that there cannot be an interior $\theta(\eta)$ solving (6) with equality. Plugging in $\hat{\theta} = 1/4$, (6) becomes

$$\frac{32\theta^5}{25(2-\eta)^2} - 8\theta^3 + \theta + \frac{\theta}{10(2-\eta)} - \frac{1}{32} - \frac{1}{160(2-\eta)} - \frac{1}{3200(2-\eta)^2} = 0.$$

The left hand side of the last equation is strictly concave on the relevant range for $\eta$. Along the lines of the argument in the previous paragraph, this is sufficient for showing that there is no interior $\Phi^\eta$ minimizer.

*WA 1.2. Non-local constraints binding in the interior*

This example deals with optimal non-linear pricing by a monopolist. Its main purpose is to illustrate how to use the algorithm to solve the problem numerically in the more complicated case in which non-local incentive constraints are binding from an interval of interior types. Roughly speaking, the private information of the consumers is their demand elasticity.[24] More specifically, their utility function is

$$v(q,\theta) = q^\theta + \frac{3}{10}\theta.$$

The cross-derivative $v_{q\theta} = q^{\theta-1}(1+\theta ln(q))$ changes sign at $q = e^{-1/\theta}$. Hence, marginal utility is increasing in type for quantities $q > e^{-1/\theta}$ but decreasing in type for $q < e^{-1/\theta}$, i.e. SC is violated. Types are uniformly distributed over $[0.45, 0.95]$. The monopolist has constant marginal costs of 0.96. In this example, (CVR) does not hold everywhere. However, it is easy to verify numerically that discontinuous jumps are irrelevant.[25] The relevant first-order condition for using the algorithm is

$$2\theta q^{\theta-1} - 2*0.96 - (1-2(\theta-0.45)-\eta)(\theta q^{\theta-1}ln(q) + q^{\theta-1}) = 0.$$

Although the problem is too complicated for a algebraic solution, solutions can be calculated numerically. I describe briefly how this is done and refer to a Python file where the computation for this example is done.

For a given $\eta$, the equation can be solved for a fine grid of types. The rent function under $q(\eta,\theta)$ can be approximated using the types on the grid for which the decision is calculated: As the slope of the rent

[24] The price elasticity of demand is here defined as the relative demand change caused by a 1% increase in the marginal price. Using the first-order condition $p'(q) = \theta q^{\theta-1}$, the elasticity for the utility function below can be derived as $|1/(\theta-1)|$.

[25] The way this is done in the Python file is that for every given type it is verified that (4) defines a monotone one-to-one relationship between $q$ and $\eta$ as long as $q < 1$. As (5) holds, there is no distortion at the top. Therefore, the numerical check and theorem 5, which states that $\eta^-(\theta') \geq \eta^+(\theta')$ at discontinuity types $\theta'$, imply that the monotonicity constraint would be violated at the highest hypothetical discontinuity type.

function is $-c_\theta$, it can be calculated at every grid type. If the grid is fine enough, one can approximate the slope of the rent function for types not on the grid using the slope of close types that are on the grid. Consequently, the rents of all types can be (approximately) calculated. This allows to write the $\Phi^\eta(\theta,\hat\theta)$ function according to (NLIC). This function can then be minimized numerically. This is done repeatedly for increasing values of $\eta$ (starting from zero) until the minimum of $\Phi^\eta(\theta,\hat\theta)$ is zero. This gives $\bar\eta$. In the process, one has obtained already the minimizing types for $\eta<\bar\theta$ between which non-local incentive constraints will bind. By theorem 1, $\eta$ is constant at all other types which allows to compute the optimal decision for those types.

In this example, the solution in figure WA.1 emerges. The non-local incentive constraints are binding from an interval of interior types to the lowest type. Non-local incentive constraints are slack for the highest types. Therefore, the optimal decision coincides with the relaxed decision for those types.

*WA 1.3. Bunching and differently signed third derivatives*

I want to discuss the assumptions on third derivatives, i.e. $c_{qq\theta}<0$ and $c_{q\theta\theta}>0$. The fact that these derivatives do not change sign ensures that the cross derivative $c_{q\theta}$ changes sign only once for any given $\theta$ (or $q$). While this property is important for the analysis (e.g. for lemma 3), it is immaterial which sign the third derivatives have as long as the sign is the same for all relevant decisions and types. To illustrate this (and also to show an example where the monotonicity constraint binds) consider the following setup: Types are distributed uniformly on $[2,3]$ and the principal's objective is the expected value of $q(\theta)-t(\theta)$. The agent's utility is given by

$$\pi(q,\theta)=t(\theta)-\frac{(q-\sigma/\theta)^2}{\theta^2}+\gamma(3-\theta).$$

The story behind this example is in the context of compensation of workers.[26] The principal is the owner of a firm and the agent a worker the firm wants to hire. For the quality of the worker, talent and effort are relevant, e.g. talent is what the worker produces in a regular working time like the 40 hours week and effort is the additional time he is willing to invest. Assume the worker creates value $q=e\theta+T$ where $T$ is his talent, $e$ is the unobservable effort and $\theta$ is his type. The owner of the firm observes a public signal, e.g. education, which is a mix of talent and productivity (he does not observe $T$ and $\theta$ directly). To be precise, assume that the signal is $\sigma=\theta*T$. Given this signal, a more productive worker will have lower talent and vice versa. The production function of the manager for a given signal is $q=e\theta+\sigma/\theta$ where $q$ is the quantity/value produced by the worker. If costs of effort are $e^2+\gamma(\theta)$ and the worker's preferences are quasilinear in money, his utility function is the one above (with $t$ denoting the wage).

Here the parameter values $\sigma=27$ and $\gamma=12$ are used. In this case, third derivatives have the following signs in the relevant range of the decision: $c_{qq\theta}<0$ and $c_{q\theta\theta}<0$. Consequently, the sign switching decision $s(\theta)$ is downward sloping. As depicted in figure WA.2a, first best decision and relaxed decision are also downward sloping.

Note that (4) is linear in $q$. It is therefore easy to verify that (4) defines a strictly monotonically one-to-one relationship between $\eta$ and $q$ (for a given $\theta$) in the relevant range. Following the arguments in the proof of proposition 3, the solution will therefore be continuous.

The example has an additional feature which is that the monotonicity constraint will be binding: $q^r$ has slope 0 at the top type $\bar\theta=3$. For $\eta>0$, $q(\cdot,\eta)$ is increasing on some interval $(\theta'(\eta),\bar\theta]$. Hence, the monotonicity constraint will have to bind for types in the top end of the type distribution if (NLIC) binds. It is easy to check that (NLIC) cannot bind to these high types (i.e. $\Phi^0(\theta,\hat\theta)>0$ for $\hat\theta$ high enough). Since $q(\cdot,\eta)$ satisfies the monotonicity constraint for all types at the lower end to which (NLIC) can bind, i.e. $\Phi^0(\bar\theta,\underline\theta)<0$ and the boundary types minimize $\Phi^0$, theorem 2 implies that (NLIC) will bind at types minimizing $\Phi^\eta$.

It turns out that also in this example $(\bar\theta,\underline\theta)$ is the unique local minimizer of $\Phi^\eta(\theta,\hat\theta)$ for all relevant $\eta\ge 0$. Therefore only the non-local incentive constraint from the highest to the lowest type is binding. However,

[26] A related example can be found in [5].

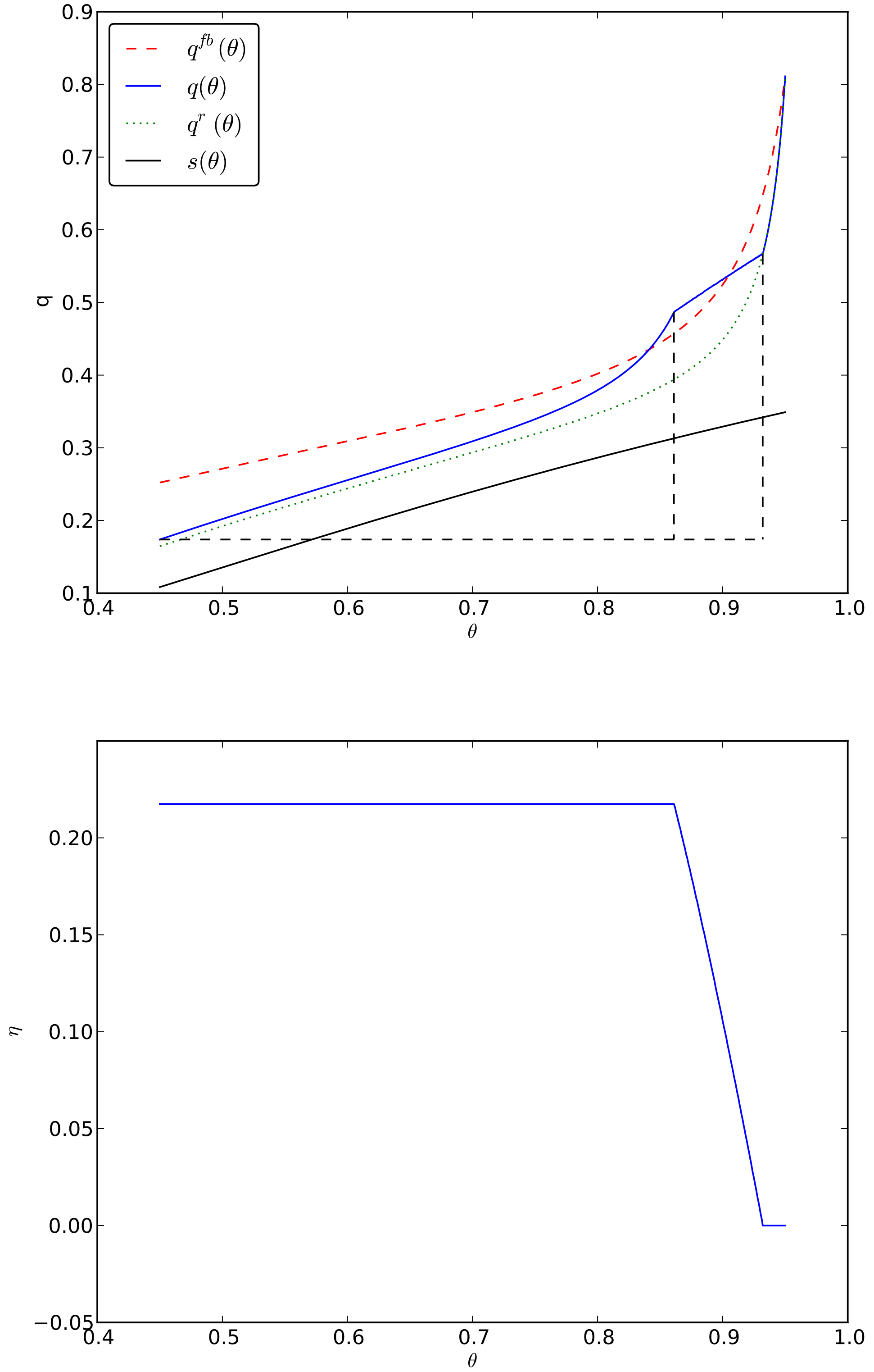


Figure WA.1: non-local constraints bind from an interval of interior types to the lowest type

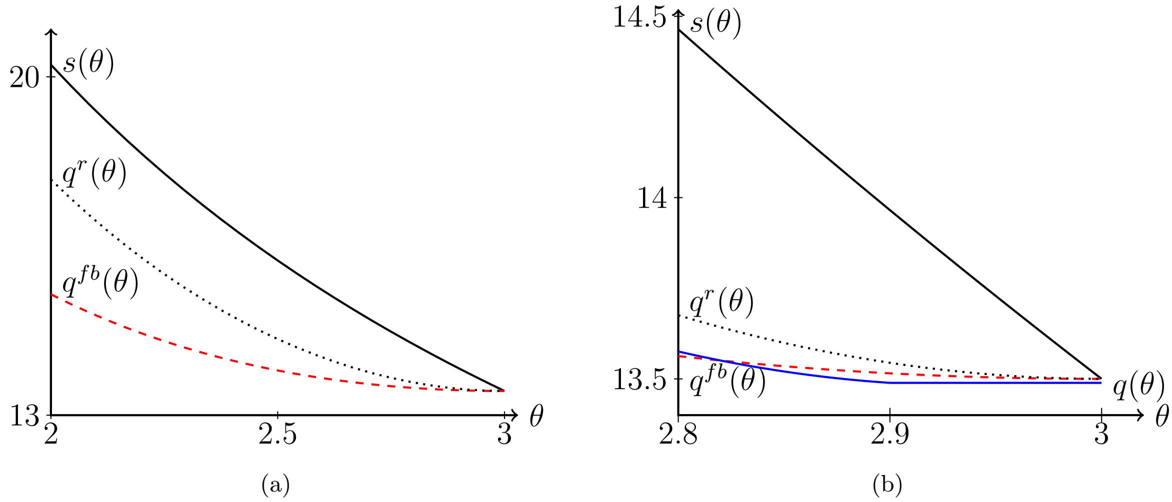


Figure WA.2: numerical example: bunching

the monotonicity constraint is binding for the highest types. For each $q(\cdot,\eta)$, the optimal bunching interval $[\theta_s(\eta),\bar{\theta}]$ is determined by the condition

$$\int_{\theta_s(\eta)}^{\bar{\theta}} [u_q(q(\theta,\eta),\theta) - c_q(q(\theta,\eta),\theta)]f(\theta) + (1-F(\theta)-\eta(\theta))c_{q\theta}(q(\theta,\eta),\theta)\, d\theta = 0.$$

For each $\eta$, this gives the bunching interval, i.e. $q(\cdot,\eta)$ is the normal $q(\cdot,\eta)$ on $[\underline{\theta},\theta_s(\eta)]$ and $q(\theta_s(\eta),\eta)$ for all types in $(\theta_s(\eta),\bar{\theta}]$. $\bar{\eta}$ is now determined such that $\Phi^{\bar{\eta}}(3,2)=0$ where $\Phi^{\eta}$ is calculated using the ironed out solution. Here, $\bar{\eta}$ turns out to be approximately 0.18 and the solution for the highest types is depicted in figure WA.2b. The solution exhibits bunching of types in $[2.9,3]$.

## WA 2. Proofs and derivations

I prove the third point of lemma 1 first as an intermediate result.

**Lemma WA.1.** *Every implementable decision $q$ is bounded from above.*

**Proof:** Take any two types $\theta'$ and $\theta''$ with $q(\theta') > s(\bar{\theta})$ and $q(\theta'') > s(\bar{\theta})$ and let $\theta'' > \theta'$.[27] Incentive compatibility implies

$$c(q(\theta''),\theta') - c(q(\theta'),\theta') \geq t(\theta'') - t(\theta') \geq c(q(\theta''),\theta'') - c(q(\theta'),\theta'').$$

Combining the two inequalities gives

$$\int_{\theta'}^{\theta''}\int_{q(\theta'')}^{q(\theta')} c_{q\theta}(q,\theta)\, dq\, d\theta \geq 0. \qquad \text{(WA.1)}$$

As $q(\theta') > s(\bar{\theta})$ and $q(\theta'') > s(\bar{\theta})$, the last inequality can only be satisfied if $q(\theta'') \geq q(\theta')$. Hence, $q$ is increasing on the set of types for which $q(\theta) > s(\bar{\theta})$. Now suppose that $q(\theta)$ was not bounded from

[27] If two such types do not exist, there is nothing to prove: Either there is exactly one type $\theta$ with $q(\theta) > s(\bar{\theta})$ which means that $q$ is bounded from above by $q(\theta)$ or $q$ is bounded from above by $s(\bar{\theta})$.

above. Then there exists exactly one limit type $\theta''$ such that for any $q' \in \mathbb{R}$ and any $\varepsilon > 0$ there exists a $\theta' \in (\theta'' - \varepsilon, \theta'')$ with $q(\theta') > q'$. Since $q(\theta'')$ is finite and $c_{qq\theta} < 0$, (WA.1) cannot hold for $\theta'$ and $\theta''$ for $q'$ high enough.[28] This contradicts that $q$ is not bounded from above. □

**Proof of lemma 1:** Boundedness of $q$ is proven in lemma WA.1. As the range of $q$ is bounded by lemma WA.1, the envelope theorem of corollary 4 in Milgrom and Segal [36] applies. Monotonicity is proven in lemma 1 of [5]. For completeness both proofs are sketched here:

Recall that $\pi(\theta) \equiv t(\theta) - c(q(\theta), \theta)$. Incentive compatibility between $\theta$ and $\hat{\theta} < \theta$ implies

$$\frac{c(q(\theta), \hat{\theta}) - c(q(\theta), \theta)}{\theta - \hat{\theta}} \geq \frac{\pi(\theta) - \pi(\hat{\theta})}{\theta - \hat{\theta}} \geq \frac{c(q(\hat{\theta}), \hat{\theta}) - c(q(\hat{\theta}), \theta)}{\theta - \hat{\theta}}.$$

As $q$ is bounded and $c$ is $C^3$, the inequalities imply that $\pi$ is a Lipschitz function. It is, therefore, almost everywhere differentiable with $\pi_\theta(\theta) = -c_\theta(q(\theta), \theta)$ for almost all types. The envelope condition follows by the fundamental theorem of calculus.

To show monotonicity, take a $\theta'$ with $q(\theta') > s(\theta')$ and let $q$ be right continuous with left-hand side limit at $\theta'$. By right continuity, there exists a type $\theta''$ such that $c_{q\theta}(q(\theta_1), \theta_2) < 0$ for all $\theta_1, \theta_2 \in (\theta', \theta'')$.

First, it is shown that $q^-(\theta') \leq q^+(\theta')$. Suppose otherwise. Then – by right continuity of $q$ at $\theta'$ – there exists a $\delta > 0$ such that $q(\theta) < q^-(\theta')$ for all $\theta \in (\theta', \theta' + \delta)$. But then $\Phi(\theta' + \delta, \theta' - \varepsilon) < 0$ for some $\varepsilon > 0$. This contradicts NLIC and proves the claim $q^-(\theta') \leq q^+(\theta')$.

Second, $q$ is increasing on $(\theta', \theta'')$. This follows directly from (WA.1) and $c_{q\theta}(q(\theta_1), \theta_2) < 0$ for all $\theta_1, \theta_2 \in (\theta', \theta'')$.

To complete the monotonicity proof, now take a $\theta'$ such that $q(\theta') < s(\theta')$ and let $q$ be right continuous with left-hand side limit at $\theta'$. By right continuity, there exists a type $\theta'' > \theta'$ such that $c_{q\theta}(q(\theta_1), \theta_2) > 0$ for all $\theta_1, \theta_2 \in (\theta', \theta'')$.

First, it is shown that $q^-(\theta') \geq q^+(\theta')$. Suppose otherwise. Then – by right continuity – there exists a $\delta > 0$ such that $q(\theta) > q^-(\theta')$ for all $\theta \in (\theta', \theta' + \delta)$. But then $\Phi(\theta' + \delta, \theta' - \varepsilon) < 0$ for some $\varepsilon > 0$ which contradicts (NLIC).

Second, $q$ is decreasing on $(\theta', \theta'')$. This follows directly from (WA.1) and $c_{q\theta}(q(\theta_1), \theta_2) > 0$ for all $\theta_1, \theta_2 \in (\theta', \theta'')$. □

**Proof of lemma 2:** By $u_{qq} - c_{qq} < 0$, $q^{fb}$ is implicitly defined by the first-order condition $u_q(q, \theta) - c_q(q, \theta) = 0$. The implicit function theorem shows that $sgn\{q_\theta^{fb}(\theta)\} = sgn\{u_{q\theta}(q^{fb}(\theta), \theta) - c_{q\theta}(q^{fb}(\theta), \theta)\}$. Assumption 2 implies that $c_{q\theta}(q^{fb}(\theta), \theta) < 0$ and therefore $q_\theta^{fb}(\theta) > 0$ for all $\theta \in \Theta$.

The concavity of the relaxed program follows directly from assumption 1 as $u_{qq} - c_{qq} + (1 - F)c_{qq\theta} < 0$. Therefore, $q^r$ is defined by the first-order condition (3). We immediately get $q^r(\bar{\theta}) = q^{fb}(\bar{\theta}) > s(\bar{\theta})$ where the inequality follows from assumption 2. The first-order condition (3) implies that no type can have $q^r(\theta) = s(\theta)$: If this was the case, (3) would imply that $q^r(\theta) = q^{fb}(\theta)$ but this would contradict assumption 2 which says $q^{fb}(\theta) > s(\theta)$. As the continuity and differentiability assumptions on $u$ and $c$ imply that $q^r$ is continuous, we get that $q^r(\bar{\theta}) > s(\bar{\theta})$ implies $q^r(\theta) > s(\theta)$ for all types. Therefore, $c_{q\theta}(q^r(\theta), \theta) < 0$ and $q^r \leq q^{fb}(\theta)$ by (3) where the inequality is strict for all types but $\bar{\theta}$.

Using the implicit function theorem on (3) yields together with assumption 1 that $q_\theta^r > 0$. □

*WA 2.1. Existence of an optimal contract menu*

This subsection shows that an optimal contract menu exists and therefore the characterization done in the paper is meaningful. Before showing existence, one intermediate result is derived which implies that the optimal decision is bounded from above by a specific value $\bar{q}$ defined below.

Define $\tilde{q} > s(\bar{\theta})$ such that $\int_0^{\tilde{q}} c_{q\theta}(y, \bar{\theta})\, dy = 0$. As $\tilde{q} > s(\bar{\theta})$, the integral is strictly decreasing in $\tilde{q}$. By $c_{qq\theta} < 0$, the integral is concave in $\tilde{q}$ which implies $\lim_{\tilde{q} \to \infty} \int_0^{\tilde{q}} c_{q\theta}(y, \bar{\theta})\, dy = -\infty$. Hence, $\tilde{q}$ exists and is unique and therefore properly defined.

[28]This argument does not require $q(\theta'')$ to be above $s(\bar{\theta})$: Because of $c_{qq\theta} < 0$, $\lim_{q' \to \infty} \int_{q(\theta'')}^{q'} c_{q\theta}(q, \theta)\, dq = -\infty$.

**Lemma WA.2.** *Any incentive compatible contract menu with a decision $q(\theta)$ above $\bar{q} = max\{q^{fb}(\bar{\theta}), \tilde{q}\}$ for some type $\theta < \bar{\theta}$ is dominated by a contract menu consisting of decision*

$$q^c(\theta) = min\{q(\theta), \bar{q}\}$$

*and transfers such that $\pi(\underline{\theta}) = 0$ and $\pi_\theta(\theta) = \int_{\underline{\theta}}^{\theta} -c_\theta(q^c(x), x)\, dx$.*

**Proof.** By lemma WA.1, the optimal decision is bounded. The boundedness of $q$ and the assumption that $c$ is three times continuously differentiable imply that the envelope theorem in [36] applies; see lemma 1. The principal's problem can, therefore, be written as maximization of the virtual valuation subject to NLIC.

Note that incentive compatibility of $q^c(\theta)$ is obvious if $q(\theta) > \bar{q}$ for all $\theta$. Now define $\theta^m = inf\{\theta : q(\theta) > \bar{q}\}$. Note that incentive compatibility from $\theta^m$ to any lower type is not affected by the change from $q$ to $q^c$ since $\Phi(\theta^m, \hat{\theta})$ does not change.

The next step is to see that $q(\theta) > \bar{q}$ for all $\theta > \theta^m$. By way of contradiction, suppose there exists a type $\theta'' > \theta^m$ with $q(\theta'') \leq \bar{q}$. By the definition of $\theta^m$, there exists a type $\theta' \in [\theta^m, \theta'')$ with $q(\theta') > \bar{q}$. By the definition of $\bar{q}$ and $c_{q\theta\theta} > 0$, (WA.1) is violated for $\theta'$ and $\theta''$: $\int_{q(\theta'')}^{q(\theta')} c_{q\theta}(y, \theta)\, dy < 0$ for any $\theta$ as $0 \leq q(\theta'') < \bar{q} < q(\theta')$. This contradicts that $q$ is incentive compatible. Consequently, $q(\theta) > \bar{q}$ for all $\theta > \theta^m$.

Therefore, all types above $\theta^m$ will have $\bar{q}$ as their changed decision. It follows from lemma 3 that only incentive compatibility from types above $\theta^m$ to types below $\theta^m$ has to be checked. Take an arbitrary $\theta > \theta^m$ and some $\hat{\theta} < \theta^m$. Then $\Phi(\theta, \hat{\theta}) = \Phi(\theta^m, \hat{\theta}) - \int_{\theta^m}^{\theta} \int_{q(\hat{\theta})}^{\bar{q}} c_{q\theta}(y, x)\, dy\, dx > 0$ where the inequality follows from the incentive compatibility between $\theta^m$ and $\hat{\theta}$ under $q(\theta)$, the definition of $\tilde{q}$ and $c_{q\theta\theta} > 0$.

The concavity of the virtual valuation implies that the principal's payoff under $q^c(\theta)$ is higher than under $q(\theta)$. In fact, it is strictly higher if $\theta^m < \bar{\theta}$ because the set of types with a changed decision has measure $1 - F(\theta^m) > 0$ in this case. □

Next, the existence proof. It is assumed that $q^v(q, \theta) \geq q^s(q, \theta)$ for all $q \in [0, s(\theta)]$ and all $\theta \in [\underline{\theta}, \bar{\theta}]$ and therefore every $q$ that is not increasing is dominated by a decision that is increasing (see the first part of the proof of proposition 2 in the main text). Given this and the previous two results, the existence proof in [23] applies. For completeness, I replicate the proof briefly. The problem of the principal is the program

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} (u(q(\theta), \theta) - c(q(\theta), \theta))f(\theta) + (1 - F(\theta))c_\theta(q(\theta), \theta)\, d\theta$$

subject to

$$\begin{aligned} \Phi(\theta, \hat{\theta}) &\geq 0 \quad \text{for all} \quad \theta, \hat{\theta} \in [\underline{\theta}, \bar{\theta}] \\ 0 &\leq q(\theta) \leq \bar{q} \quad \text{for all} \quad \theta \in [\underline{\theta}, \bar{\theta}]. \end{aligned}$$

Let $W^*$ be the supremum value of the program. Take a sequence of decision functions such that $q^n$ leads to an objective value larger than $W^* - \frac{1}{n}$ and each $q^n$ is implementable. Because of (9) and the argument there, the sequence can be chosen such that each $q^n$ is an increasing function. Because of lemma WA.2, each $q^n$ can be chosen such that $q^n \leq \bar{q}$. Then Helly's selection theorem, see Billingsley [37] Thm. 25.9, yields that there exists a increasing function $q$ which is the limit of a subsequence $q^{n_k}$ at every point of continuity of $q$ and therefore almost everywhere on $[\underline{\theta}, \bar{\theta}]$. Lebesgue's dominated convergence theorem, see [37] Thm. 16.4, yields that the principal's payoff under $q$ is $W^*$. By lemma WA.3 below, $q$ is implementable and therefore an optimal contract menu exists.

**Lemma WA.3.** *Take a sequence of implementable, increasing decision functions $q^n \leq \bar{q}$, $n = 1, 2 \ldots$. Let $q$ be an increasing decision such that $q$ is the pointwise limit of the sequence $(q^n)_{n=1}^{\infty}$ at all points of continuity of $q$. Then, $q$ is implementable.*

**Proof.** As $q$ is increasing, only downward incentive compatibility has to be checked; see lemma 3. Take two arbitrary types $\hat{\theta}$ and $\theta$ such that $\hat{\theta} < \theta$.

First, assume that $q$ is continuous at $\hat{\theta}$. Define $\tilde{c}_{q\theta} = \max_{q\in[0,q(\bar{\theta})+1],\, \theta\in[\underline{\theta},\bar{\theta}]} |c_{q\theta}(q,\theta)|$. Since $[0, q(\bar{\theta}) + 1] \times [\underline{\theta}, \bar{\theta}]$ is compact and $c_{q\theta}(\cdot)$ is continuous by assumption, $\tilde{c}_{q\theta}$ exists.

Now suppose contrary to the lemma that $\Phi(\theta, \hat{\theta}) = -\varepsilon$ for and some $\varepsilon > 0$ and therefore incentive compatibility is violated under $q$. From convergence of $(q^n)_{n=1}^{\infty}$, for each $\delta > 0$ there exists an $N_\delta$ such that $|q^n(\theta) - q(\theta)| \leq \delta$ for all types at which $q$ is continuous and all $n > N_\delta$. Note that by the monotonicity of $q$, $q$ is continuous almost everywhere. Therefore,

$$\Phi(\theta,\hat{\theta}) = \int_{\hat{\theta}}^{\theta}\int_{q(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx \geq \int_{\hat{\theta}}^{\theta}\int_{q^n(\hat{\theta})}^{q^n(x)} -c_{q\theta}(y,x)\,dy\,dx - \int_{\hat{\theta}}^{\theta} 2\delta\tilde{c}_{q\theta}\,dx$$

for an arbitrary $n > N_\delta$. But then choosing a $\delta < \varepsilon/[2\tilde{c}_{q\theta}(\bar{\theta} - \underline{\theta})]$ shows that $\Phi(\theta, \hat{\theta}) > -\varepsilon$ as $\Phi^{q^n}(\theta, \hat{\theta}) \geq 0$ where $\Phi^{q^n}$ denotes $\Phi$ under $q^n$. This contradicts the definition of $\varepsilon$ and therefore $q(\theta)$ is incentive compatible.

Second, let $q$ be discontinuous at $\hat{\theta}$. There are two cases: Either (i) $\int_{\hat{\theta}}^{\theta} c_{q\theta}(q(\hat{\theta}), x)\,dx \geq 0$ or (ii) $\int_{\hat{\theta}}^{\theta} c_{q\theta}(q(\hat{\theta}), x)\,dx < 0$. If (i), then

$$\Phi(\theta,\hat{\theta}) = \int_{\hat{\theta}}^{\theta}\int_{q(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx \geq \int_{\hat{\theta}}^{\theta}\int_{q^-(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx$$

because of (i) and $c_{qq\theta} < 0$. By the first step, incentive compatibility is satisfied for all types at which $q$ is continuous and therefore

$$\int_{\hat{\theta}}^{\theta}\int_{q^-(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx \geq 0$$

which then implies $\Phi(\theta, \hat{\theta}) \geq 0$.

If (ii), then

$$\Phi(\theta,\hat{\theta}) = \int_{\hat{\theta}}^{\theta}\int_{q(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx \geq \int_{\hat{\theta}}^{\theta}\int_{q^+(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx$$

because of (ii) and $c_{qq\theta} < 0$. As $q$ is incentive compatible at all points of continuity of $q$, the last inequality implies $\Phi(\theta, \hat{\theta}) \geq 0$. □

*WA 2.2. Addenda proof theorem 1*

*Skipped cases when showing that $\eta$ is decreasing at discontinuity type $\theta'$.* The main text skipped the proofs of incentive compatibility in cases (ii) and (iii). Both are provided here. What remains to be shown is that the changed decision is incentive compatible in these two cases. Case (ii) is considered first.

First, take $\hat{\theta} < \theta'$. As described in the main text, the continuity of $\Phi^c$ in $\varepsilon'$ implies that it is sufficient to show that $\Phi(\theta, \hat{\theta}) > 0$ for all $\hat{\theta} \in (\theta' - \varepsilon, \theta')$ and $\theta \geq \theta'$ for some $\varepsilon > 0$. Suppose, contrary to this, that there are types $\theta$ such that $\Phi(\theta, \hat{\theta}) = 0$ for $\hat{\theta} < \theta'$ arbitrarily close to $\theta'$. Then there exists a type $\theta > \theta'$ that is indifferent between his contract and the limit contract associated with $q^-(\theta')$. This implies $\int_{\theta'}^{\theta} c_{q\theta}(q^-(\theta'), x)\,dx \leq 0$ as otherwise $\Phi(\theta, \theta' - \varepsilon) < 0$ which would violate incentive compatibility. But then $\Phi(\theta, \theta') = \Phi(\theta, \theta'^-) + \int_{\theta'}^{\theta}\int_{q^-(\theta')}^{q(\theta')} c_{q\theta}(y, x)\,dy\,dx < 0$ by $c_{qq\theta} < 0$ and $\int_{\theta'}^{\theta} c_{q\theta}(q^-(\theta'), x)\,dx \leq 0$. Hence, for $\varepsilon > 0$ small enough $\Phi(\theta, \hat{\theta}) > 0$ for all $\theta \neq \hat{\theta}$ and $\hat{\theta} \in (\theta' - \varepsilon, \theta')$.

Second, take $\hat{\theta} \in [\theta', \theta' + \varepsilon)$. Choose $\varepsilon > 0$ small enough such that $q(\theta) = q(\theta')$ for all $\theta \in [\theta', \theta' + \varepsilon)$. Suppose that $\Phi(\theta, \theta') = 0$ for some $\theta$ with $q(\theta) > q(\theta')$. Note that $\int_{\theta'}^{\theta} c_{q\theta}(q(\theta'), x)\,dx < 0$ as otherwise $\Phi(\theta, \theta' - \varepsilon'') \leq \Phi(\theta, \theta') - \int_{\theta'}^{\theta}\int_{q^-(\theta')}^{q^+(\theta')} c_{q\theta}(y, x)\,dy\,dx < 0$ for $\varepsilon'' > 0$ small enough. Now let $\varepsilon > 0$ be small enough such that $\int_{\theta'+\varepsilon}^{\theta} c_{q\theta}(q(\theta'), x)\,dx < 0$. The changed contract for type $\hat{\theta} \in [\theta', \theta' + \varepsilon)$ is then less attractive for $\theta$: $\Phi^c(\theta, \hat{\theta}) = \Phi(\theta, \theta') - \int_{\hat{\theta}}^{\theta}\int_{q^c(\hat{\theta})}^{min(q^c(x), q(\theta'))} c_{q\theta}(y, x)\,dy\,dx > 0$ for $\varepsilon' > 0$ small enough because

$\int_{\theta'+\varepsilon}^{\theta} c_{q\theta}(q(\theta'),x)\,dx < 0$ implies $\int_{\theta'+\varepsilon}^{\theta} c_{q\theta}(y,x)\,dx < 0$ for $y \in [q^c(\hat\theta), min(q^c(x), q(\theta'))]$. Hence, the change relaxes (NLIC). If there is no type $\theta$ such that $\Phi(\theta,\theta') = 0$, then – by the definition of case (ii) and the result of the previous paragraph – there is an open neighborhood of $\theta'$ such that $\Phi(\theta,\hat\theta) > 0$ for all $\hat\theta$ in this neighborhood and $\theta$ such that $q(\theta) \neq q(\hat\theta)$. This implies that $q^c$ is incentive compatible for $\varepsilon' > 0$ small enough (as shown in the main text).

Case (iii) is a combination of case (i) and case (ii). Choose $\varepsilon > 0$ small enough such that $q(\theta' - \varepsilon) = q^-(\theta')$ and $q(\theta') = q(\theta' + \varepsilon)$. Analogous to the two cases it is then shown that the change relaxes those non-local incentive constraints that could hold with equality. If there is a type $\theta$ such that $\Phi(\theta,\theta') = 0$, then $\int_{\theta'}^{\theta} c_{q\theta}(q(\theta'),x)\,dx < 0$ as shown in case (ii). Taking $\varepsilon > 0$ small enough such that $\int_{\theta'+\varepsilon}^{\theta} c_{q\theta}(q(\theta'),x)\,dx < 0$. Then for $\hat\theta \in [\theta',\theta'+\varepsilon)$ the incentive constraint is relaxed: $\Phi^c(\theta,\hat\theta) = \Phi(\theta,\theta') - \int_{\hat\theta}^{\theta}\int_{q^c(\hat\theta)}^{min(q^c(x),q(\theta'))} c_{q\theta}(y,x)\,dy\,dx > 0$ for $\varepsilon' > 0$ small enough because $\int_{\theta'+\varepsilon}^{\theta} c_{q\theta}(q(\theta'),x)\,dx < 0$ implies $\int_{\theta'+\varepsilon}^{\theta} c_{q\theta}(y,x)\,dx < 0$ for $y \in [q^c(\hat\theta), q(\theta')]$.

Analogously to case (i), say there is a type $\theta > \theta'$ such that $\Phi(\theta,\hat\theta) = 0$ for $\hat\theta \in (\theta' - \varepsilon, \theta')$. Then $\int_{\theta'}^{\theta} c_{q\theta}(q^-(\theta'),x)\,dx > 0$ as shown in case (i) and we can choose $\varepsilon > 0$ small enough such that $\int_{\theta'-\varepsilon}^{\theta} c_{q\theta}(q^-(\theta'),x)\,dx > 0$. The changed contract of types $\hat\theta \in (\theta'-\varepsilon,\theta')$ is then less attractive for $\theta$: $\Phi^c(\theta,\hat\theta) = \Phi(\theta,\theta'-\varepsilon) + \int_{\theta'-\varepsilon}^{\theta}\int_{q(\theta'-\varepsilon)}^{min(q^c(x),q^c(\hat\theta))} c_{q\theta}(y,x)\,dy\,dx > 0$ for $\varepsilon' > 0$ small enough as $\int_{\theta'-\varepsilon}^{\theta} c_{q\theta}(q^-(\theta'),x)\,dx > 0$ implies $\int_{\theta'-\varepsilon}^{\theta} c_{q\theta}(y,x)\,dx > 0$ for $y \in (q(\theta'-\varepsilon), min(q^c(x), q^c(\hat\theta)))$.

*WA 2.3. Binding monotonicity constraint*

This subsection shows how the properties of theorem 1 extend if the monotonicity constraint is explicitly taken into account. Instead of (4) the solution is then characterized by

$$\nu_\theta(\theta) = (u_q(q(\theta),\theta) - c_q(q(\theta),\theta))f(\theta) + (1 - F(\theta) - \eta(\theta))c_{q\theta}(q(\theta),\theta) \tag{WA.2}$$

where $\nu(\theta)q_\theta(\theta) = 0$ for all $\theta \in \Theta$, i.e. $\nu(\theta)$ is the Lagrange parameter of the monotonicity constraint. If the start and ending type of an interval on which the monotonicity constraint is binding are denoted by $\theta_s^b$ and $\theta_e^b$, then obviously $\int_{\theta_s^b}^{\theta_e^b} \nu_\theta(\theta)\,d\theta = 0$. As described in the existing literature on ironing, see [38] or the exposition in [21, ch. 7], the interval is characterized by this last condition and the endpoint conditions $\nu(\theta_s^b) = \nu(\theta_e^b) = 0$. Lemma WA.4 formally shows that $\eta$ in (WA.2) has the properties of theorem 1 also for types at which the monotonicity constraint binds.

**Lemma WA.4.** *If the monotonicity constraint binds for types in $(\theta_s^b,\theta_e^b)$ in the optimal solution, then there exists a function $\eta(\theta)$ which satisfies the properties of theorem 1 also for types in $(\theta_s^b,\theta_e^b)$. In particular, $\eta$ is increasing on $(\theta_s^b,\theta_e^b)$ and constant if $\Phi(\theta,\hat\theta) > 0$ for all $\hat\theta \in (\theta_s^b,\theta_e^b)$ and $\theta > \theta_e^b$. Furthermore, $\eta$ satisfies (i) $\eta(\theta) = \eta(\hat\theta)$ if $\Phi(\theta,\hat\theta) = 0$ and (C1) as well as (C2) hold, (ii) $\int_{\theta_s^b}^{\theta_e^b} \nu_\theta(\theta)\,d\theta = 0$ with $\nu_\theta(\theta)$ defined as in (WA.2).*

**Proof.** From lemma 4, $\Phi(\theta,\hat\theta) > 0$ for all $\theta \in [\theta_s^b,\theta_e^b)$ and $\hat\theta < \theta_s^b$. To satisfy similar properties as in theorem 1, $\eta(\theta)$ has therefore to be increasing on $(\theta_s^b,\theta_e^b)$.

Let $\eta$ be defined by (4) for all types not in $(\theta_s^b,\theta_e^b)$. Define $\eta$ for types in $(\theta_s^b,\theta_e^b)$ using the following two step procedure: First, all $\hat\theta \in (\theta_s^b,\theta_e^b)$ such that there exists a $\theta$ with $\Phi(\theta,\hat\theta) = 0$ and (C1) as well as (C2) are satisfied are assigned $\eta(\hat\theta) = \eta(\theta)$. Second, types in $\theta \in (\theta_s^b,\theta_e^b)$ who are not assigned a value for $\eta(\theta)$ in step 1 are assigned the same $\eta$ as the highest type $\theta' < \theta$ that was already assigned a value $\eta(\theta')$.

Now it is shown that the constructed $\eta$ is increasing on $(\theta_s^b,\theta_e^b)$: Say, there are two types $\hat\theta_1,\hat\theta_2 \in (\theta_s^b,\theta_e^b)$ with $\hat\theta_2 > \hat\theta_1$ which are assigned an $\eta$ in the first step. Then (C2) implies that $\theta_1 > \theta_2$. From theorem 1 and the structure of the solution as depicted in figure 4, it follows that $\eta(\theta_2) \geq \eta(\theta_1)$. Therefore, $\eta(\hat\theta_2) \geq \eta(\hat\theta_1)$. The second step does not change the monotonicity of $\eta(\theta)$ which proves that $\eta(\theta)$ is increasing on $(\theta_s^b,\theta_e^b)$.

If $\Phi(\theta,\hat\theta) > 0$ for all $\hat\theta \in (\theta_s^b,\theta_e^b)$ and $\theta > \theta_e^b$, no type is assigned a value for $\eta$ in step 1. Consequently, $\eta$ is constant on $(\theta_s^b,\theta_e^b)$.

Last, it is necessary to show that – with the above constructed $\eta$ on $(\theta_s^b, \theta_e^b)$ – there is no upward jump of $\eta$ at $\theta_e^b$. If no type is assigned an $\eta$ in the first step of the procedure above, this is obvious from theorem 1. Therefore, take the case where some type in $(\theta_s^b, \theta_e^b)$ is assigned a value $\eta(\theta)$ in the first step of the procedure. Then the claim follows from theorem 1: Say, $\eta^-(\theta_e^b) = \eta(\theta_1)$ for some type $\theta_1$ such that $\Phi(\theta_1, \hat{\theta}) = 0$ for $\hat{\theta} \in (\theta_s^b, \theta_e^b]$. Lemma 5 and theorem 1 imply that $\eta^+(\theta_e^b) = \eta^-(\theta_1)$.[29] Since $\eta$ is decreasing[30] at $\theta_1$ according to theorem 1, it follows that $\eta^-(\theta_1) \geq \eta^+(\theta_1)$ and therefore $\eta^-(\theta_e^b) \geq \eta^+(\theta_e^b)$. A similar argument holds for $\theta_s^b$.

It remains to show $\int_{\theta_s^b}^{\theta_e^b} \nu_\theta(\theta)\, d\theta = 0$. But this follows directly from $\nu(\theta_s^b) = \nu(\theta_e^b) = 0$. □

*WA 2.4. Stochastic contracts*

The paper concentrated on deterministic contracts. Although hardly observed in practice, one could think of stochastic contracts. In the framework of the paper, this would mean that a type $\theta$ is assigned a probability distribution over the decision $q$ instead of one deterministic decision $q(\theta)$. The idea behind a stochastic contract is to relax incentive constraints. Intuitively, this could work if different types have different degrees of risk aversion. See Rochet [16] for an example where random contracts are optimal. The following proposition gives a sufficient condition under which deterministic decisions are optimal.[31]

**Proposition WA.1.** *The optimal decision is deterministic if the assumptions of proposition 2, (CVR) and*

$$\frac{\partial \frac{c_{q\theta\theta}}{c_{q\theta}}}{\partial q} \geq 0 \tag{WA.3}$$

*hold.*

Condition (CVR) and (WA.3) differ from the conditions for non-stochastic contracts in Maskin and Riley [39]. There, only local incentive constraints bind and they bind "downward". It is then shown that assigning the expected decision increases the principal's payoff and relaxes local incentive constraints if risk aversion is decreasing in type. Decreasing risk aversion is therefore a sufficient condition for the optimality of deterministic contracts. This reasoning is flawed in case non-local incentive constraints are binding: Assigning the expected decision decreases the slope of the rent function $\pi(\theta)$ because $-c_\theta$ is convex in $q$. Hence, profit differences between $\theta$ and $\hat{\theta}$ are smaller under the expected decision compared to the stochastic contract. Non-local incentive constraints are, therefore, harder to satisfy.

Strausz [40] points out that stochastic mechanisms are not optimal whenever the principal's problem can be written as a pointwise maximization of the virtual valuation. However, this is not the case if SC is violated because the virtual valuation maximization is subject to (NLIC).

Proposition WA.1 takes therefore another way known from Jullien [23]. When rewriting the principal's optimization problem as an optimization over rent profiles $\pi$ (instead of over decision functions $q$) condition (CVR) ensures that the resulting program is concave. Condition (WA.3) ensures that the set of implementable utility profiles is convex. These two properties imply that a stochastic decision is worse for the principal than a deterministic decision implementing the same utility profile. The conditions of proposition 2 allow to focus on decisions above $s$, i.e. monotone solutions.

**Proof of proposition WA.1.** First, note that under the conditions of proposition 2 one can focus on decisions above $s(\theta)$: If some $q(\theta) < s(\theta)$ was used in a stochastic contract menu with positive probability,

[29] If non-local incentive constraints bind from types $\theta' \in (\theta_e^b, \theta_1)$ to types $\hat{\theta}' \in (\theta_e^b, \theta_1)$, this holds still true because of the necessary condition (C3). Also discontinuities at $\theta'' \in (\theta_e^b, \theta_1)$ do not matter as by lemma 4 and theorem 1 $\eta(\theta)$ is decreasing at $\theta''$. If there are several bunching intervals, the argument holds for the highest interval and given this, it holds for the second highest etc..

[30] Note that $q$ is strictly increasing at $\theta_1$ if it is continuous there (lemma 4). If $q$ is discontinuous at $\theta_1$, theorem 1 states that $\eta^-(\theta_1) \geq \eta^+(\theta_1)$.

[31] To illustrate: In example 1, $\frac{\partial \frac{c_{q\theta\theta}}{c_{q\theta}}}{\partial q} = \frac{4\theta^3}{(\theta^3 - 2q\theta)^2} \geq 0$ and condition (WA.3) holds.

the principal could do better by assigning $q^s(q(\theta),\theta)$ instead of $q(\theta)$. The proof is equivalent to the one of proposition 2.[32]

Second, suppose the optimal contract menu was stochastic and denote by $G(q,\theta)$ the distribution of $q$ at type $\theta$. Consider now an alternative deterministic contract menu $q^*(\theta)$ where $q^*(\theta) \geq s(\theta)$ is determined such that $c_\theta(q^*(\theta),\theta) = \int_q c_\theta(q,\theta)\,dG(q,\theta)$. In short, the slope of the rent function $\pi(\theta)$ and therefore the rent of each type remains the same under both menus.[33] It will be shown that under the assumptions of proposition WA.1 this change increases the principal's payoff and relaxes incentive compatibility.

Since only $q(\theta) \geq s(\theta)$ have to be considered, there is a one to one relationship between $q$ and $-c_\theta(q,\theta)$. Define $h(z,\theta) \geq s(\theta)$ as the decision corresponding to $-c_\theta$ being $z$, i.e. $z = -c_\theta(h(z,\theta),\theta)$. Then the principal's objective can be written as

$$W = \int_{\underline{\theta}}^{\bar{\theta}} \left([u(h(z,\theta),\theta) - c(h(z,\theta),\theta)]f(\theta) - [1-F(\theta)]z\right)\, d\theta. \tag{WA.4}$$

The next step is to show that $W$ is concave in $z$. This implies that the deterministic decision $q^*$ increases the principal's payoff. The last step will then be to show that this deterministic decision is also incentive compatible.

Using $h_z(z,\theta) = \frac{1}{-c_{q\theta}(h,\theta)}$, which follows from the definition of $h(z,\theta)$, it is straightforward to derive

$$\frac{\partial^2 W}{\partial z^2} = \int_{\underline{\theta}}^{\bar{\theta}} \left( \frac{c_{qq\theta}(h,\theta)}{c_{q\theta}^2(h,\theta)} \left[ \frac{u_{qq}(h,\theta) - c_{qq}(h,\theta)}{c_{qq\theta}} - \frac{u_q(h,\theta) - c_q(h,\theta)}{c_{q\theta}(h,\theta)} \right] f(\theta) \right) d\theta.$$

By condition (CVR), the integrand is negative and therefore $W$ is concave in $z$.

Incentive compatibility of $q^*$ means that for arbitrary types $\theta$ and $\hat{\theta}$

$$\Phi^*(\theta,\hat{\theta}) \equiv \int_{\hat{\theta}}^{\theta} \left( c_\theta(q^*(\hat{\theta}),x) - c_\theta(q^*(x),x) \right) dx \geq 0.$$

To verify this, it is useful to see that (WA.3) implies that $-c_{\theta\theta}(h(z,\theta),\theta)$ is convex in $z$:

$$\frac{d^2\{-c_{\theta\theta}(h(z,\theta),\theta)\}}{dz^2} = \frac{-1}{c_{q\theta}} \frac{\partial \frac{c_{q\theta\theta}}{c_{q\theta}}}{\partial q} \geq 0.$$

This convexity implies

$$\frac{d}{d\theta} \left\{ \int_q -c_\theta(q,\theta)\, dG(q,\hat{\theta}) + c_\theta(q^*(\hat{\theta}),\theta) \right\} = \int_q -c_{\theta\theta}(q,\theta)\, dG(q,\hat{\theta}) + c_{\theta\theta}(q^*(\hat{\theta}),\theta) \geq 0.$$

The last inequality implies that $\int_{\hat{\theta}}^{\theta} \int_q -c_\theta(q,x)\, dG(q,\hat{\theta}) + c_\theta(q^*(\hat{\theta}),x)\, dx$ is a convex function of $\theta$ with a minimum at $\theta = \hat{\theta}$ where the function value is 0. Consequently,

$$\int_{\hat{\theta}}^{\theta} c_\theta(q^*(\hat{\theta}),x)\, dx \geq \int_{\hat{\theta}}^{\theta} \int_q c_\theta(q,x)\, dG(q,\hat{\theta})\, dx.$$

[32] Admittedly, it is not obvious whether lemma 3 holds for stochastic contract menus. Therefore, I show here explicitly that upward incentive constraints are relaxed if all $q(\theta) < s(\theta)$ are substituted by $q^s(q(\theta),\theta)$. Take $\hat{\theta} > \theta$, then the incentive constraint can be written as $\int_{\theta}^{\hat{\theta}} \int_q c_\theta(q,x)\, dG(q,x) - \int_q c_\theta(q,x)\, dG(q,\hat{\theta})\, dx \geq 0$. Changing $q(\theta) < s(\theta)$ to $q^s(q(\theta),\theta)$ does not change the first term. But since $\int_{q(\theta)}^{q^s(q(\theta),\theta)} c_{q\theta}(q,x)\, dq < 0$ for all $x < \hat{\theta}$, the change relaxes the incentive constraint through the second term.

[33] It is straightforward to check that local incentive compatibility requires the slope of the rent function under the stochastic menu to be $\int_q c_\theta(\tilde{q},\theta)\, dG(q,\theta)$.

But then,

$$\Phi^*(\theta,\hat\theta)=\int_{\hat\theta}^{\theta}\left(c_\theta(q^*(\hat\theta),x)-c_\theta(q^*(x),x)\right)\,dx\geq\int_{\hat\theta}^{\theta}\left(\int_q c_\theta(q,x)\,dG(q,\hat\theta)-c_\theta(q^*(x),x)\right)\,dx\geq 0$$

where the last inequality follows from the incentive compatibility of the stochastic menu $G(q,\theta)$ and the definition of $q^*$. □

*WA 2.5. Algorithm: Addenda and extensions*

This subsection consists of one addendum to the main text. Afterwards the possibility of several $\Phi^\eta$ minimizer and its implications for the algorithm are discussed.

**Addendum to the proof of theorem 2:** The proof of the main text showed that $\Phi^\eta(\theta',\hat\theta')\leq\Phi^\eta(\theta'',\hat\theta'')$ for either $\theta''\in(\theta',\theta'+\varepsilon)$ and $\hat\theta''\in(\hat\theta',\hat\theta'+\varepsilon)$ or $\theta''\in(\theta'-\varepsilon,\theta')$ and $\hat\theta''\in(\hat\theta',\hat\theta'+\varepsilon)$. Here I present the argument for the remaining two cases which is, however, very similar to the main text.

First, take $\theta''\in(\theta',\theta'+\varepsilon)$ and $\hat\theta'\in(\hat\theta'-\varepsilon,\hat\theta')$. By the continuity of $q$ and $q(\cdot,\eta')$, there exists a $\hat\theta'''\in(\hat\theta'',\hat\theta')$ such that $q(\hat\theta''')=q(\hat\theta'',\eta')$. Then,

$$\Phi^\eta(\theta'',\hat\theta'')-\Phi^\eta(\theta',\hat\theta')=\Phi(\theta'',\hat\theta''')-\Phi(\theta',\hat\theta')+\int_{\hat\theta''}^{\hat\theta'''}\int_{q(\hat\theta''')}^{q(x,\eta')}-c_{q\theta}(y,x)\,dy\,dx$$
$$+\int_{\hat\theta'''}^{\hat\theta'}\int_{q(x)}^{q(x,\eta')}-c_{q\theta}(y,x)\,dy\,dx+\int_{\theta'}^{\theta''}\int_{q(x)}^{q(x,\eta')}-c_{q\theta}(y,x)\,dy\,dx\geq\Phi(\theta'',\hat\theta''')-\Phi(\theta',\hat\theta')\geq 0$$

where the first inequality follows from $q(x)>s(x)$ and $q(x,\eta')>s(x)$. The second inequality follows from $\Phi(\theta'',\hat\theta''')\geq 0=\Phi(\theta',\hat\theta')$ by incentive compatibility and the assumption that (NLIC) binds from $\theta'$ to $\hat\theta'$.

Second, let $\theta''\in(\theta'-\varepsilon,\theta')$ and $\hat\theta'\in(\hat\theta'-\varepsilon,\hat\theta')$. By the continuity of $q$ and $q(\cdot,\eta')$, there exists a $\hat\theta'''\in(\hat\theta'',\hat\theta')$ such that $q(\hat\theta''')=q(\hat\theta'',\eta')$. Then,

$$\Phi^\eta(\theta'',\hat\theta'')-\Phi^\eta(\theta',\hat\theta')=\Phi(\theta'',\hat\theta''')-\Phi(\theta',\hat\theta')+\int_{\hat\theta''}^{\hat\theta'''}\int_{q(\hat\theta''')}^{q(x,\eta')}-c_{q\theta}(y,x)\,dy\,dx$$
$$+\int_{\hat\theta'''}^{\hat\theta'}\int_{q(x)}^{q(x,\eta')}-c_{q\theta}(y,x)\,dy\,dx-\int_{\theta''}^{\theta'}\int_{q(x,\eta')}^{q(x)}-c_{q\theta}(y,x)\,dy\,dx\geq\Phi(\theta'',\hat\theta''')-\Phi(\theta',\hat\theta')\geq 0$$

where the first inequality follows from $q(x)>s(x)$ and $q(x,\eta')>s(x)$. The second inequality follows from $\Phi(\theta'',\hat\theta''')\geq 0=\Phi(\theta',\hat\theta')$ by incentive compatibiltiy and the assumption that (NLIC) binds from $\theta'$ to $\hat\theta'$. □

*WA 2.5.1. Several $\Phi^\eta$ minimizers*

Theorem 2 states that (NLIC) binds only between minimizers of $\Phi^\eta$. If – for a given $\eta$ – there are several such minimizers it is not a priori clear which minimizer is the one where (NLIC) binds. I will go through several configurations of $\Phi^\eta$ minimizers and show that incentive compatibility puts enough structure on the problem to determine which $\Phi^\eta$ minimizer is "relevant" (in the sense that (NLIC) binds at this minimizer). These results take $\eta$ as given, that is, $\Phi^\eta$ has several minimizers for a given $\eta$ and the question is which is/are the minimizer(s) at which (NLIC) binds. While the configurations of minimizers discussed below do not cover all possible cases, they nevertheless illustrate how the results in the main text can be extended if necessary.

Throughout I keep the assumption that either (CVR) is satisfied and $q(\cdot,\eta)$ is strictly increasing for all $\eta\geq 0$ or that the conditions of proposition 4 are satisfied.

*Case 1 (separated minimizers).* Let $(\theta_i(\eta),\hat\theta_i(\eta))$, $i=1\dots n$, be the local minimizers of $\Phi^\eta$. Assume that $\hat\theta_i<\theta_i<\hat\theta_{i+1}<\theta_{i+1}$ for $i=1\dots n-1$. This case turns out to be a simple extension of the argument in the main text. The solution can consist of several "brackets" of type pairs between which (NLIC) binds; see figure 4 for an illustration.

**Lemma WA.5.** *Given the assumptions of case 1, (NLIC) binds in the solution of (SB) from $\theta_i(\eta)$ to $\hat{\theta}_i(\eta)$ if and only if $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) \leq 0$.*

Lemma WA.5 implies that the algorithm has to be adjusted in case 1 such that only those minimizers $(\theta_i(\eta), \hat{\theta}_i(\eta))$ are assigned a decision in 4a for which $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) \leq 0$. The algorithm will then assign the optimal decision to all types from or to which (NLIC) binds. By theorem 1, the algorithm then also assigns the optimal decision to all other types.

**Proof of lemma WA.5:** As (NLIC) cannot bind from $\theta_i(\eta')$ to $\hat{\theta}_i(\eta')$ if $\Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) > 0$ (theorem 2), $\Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) \leq 0$ for $i = 1, \ldots, n$ is assumed for notational convenience in this proof. Because of theorem 2, no relevant $\Phi^{\eta'}$ minimizer is lost by this assumption.

First, it is shown that $\eta(\theta) \geq \eta'$ for all $\theta \in [\hat{\theta}_i(\eta'), \theta_i(\eta')]$. To start with, suppose $\eta(\theta) < \eta'$ for all $\theta \in [\hat{\theta}_i(\eta'), \theta_i(\eta')]$. Then $q(\theta) \leq q(\theta, \eta')$ for all $\theta \in [\hat{\theta}_i(\eta'), \theta_i(\eta')]$. Let $\hat{\theta}'$ be the type where $q(\hat{\theta}') = q(\hat{\theta}_i(\eta'), \eta')$.[34] Then (NLIC) between $\theta_i(\eta')$ and $\hat{\theta}'$ is violated as $\Phi(\theta_i(\eta'), \hat{\theta}') < \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) \leq 0$. Next, suppose there are types $\theta_0$ and $\theta_1$ both in $(\hat{\theta}_i(\eta'), \theta_i(\eta'))$ such that $\eta(\theta_0) < \eta' \leq \eta(\theta_1)$. Suppose $\theta_0 < \theta_1$ and define $\theta' = sup\{\theta : \eta(\theta) < \eta' \text{ and } \theta \leq \theta_1\}$. By the continuity of $\eta$, $\theta'$ exists and $\eta(\theta') = \eta'$. (NLIC) binds to $\theta'$ because of lemma 7. Then theorem 2 implies that $(\theta'', \theta')$ are local minimizer of $\Phi^{\eta'}$ for some $\theta''$. But by the definition of case 1 this is impossible. A similar argument applies if $\theta_0 > \theta_1$, i.e. then there is a type $\theta' \in (\theta_1, \theta_0)$ with $\eta(\theta') = \eta'$ from which (NLIC) binds which is again impossible by the definition of case 1. Hence, $\eta(\theta) \geq \eta'$ for all $\theta \in [\hat{\theta}_i(\eta'), \theta_i(\eta')]$.

Second, the claim of the lemma holds for $(\theta_n(\eta'), \hat{\theta}_n(\eta'))$.

Define $\theta'_n = inf\{\theta : \eta(\theta) < \eta' \text{ and } \theta \geq \theta_n(\eta')\}$. If $\theta'_n$ exists, $\eta(\theta'_n) = \eta'$ by the continuity of $\eta$. If $\theta'_n > \theta_n(\eta')$, (NLIC) would have to bind from $\theta'_n$ (lemma 7). By theorem 2, $\theta'_n$ has then to be part of a minimizer of $\Phi^{\eta'}$. This would contradict the definition of $\theta_n$ as highest type minimizing $\Phi^{\eta'}$. Hence, if $\theta'_n$ exists, then $\theta'_n = \theta_n(\eta')$ and (NLIC) binds from $\theta'_n$ to $\hat{\theta}'_n$, i.e. the lemma holds in this case.

If $\theta'_n$ does not exist, $\eta(\bar{\theta}) \geq \eta'$ (by the first step and continuity of $\eta$) and (NLIC) binds from $\bar{\theta}$ (theorem 1). Hence, $(\bar{\theta}, \hat{\theta}'')$ is a local minimizer of $\Phi^{\eta''}$ for some $\hat{\theta}''$ and some $\eta'' \geq \eta'$ by theorem 2. If $\eta'' = \eta'$, then $\bar{\theta} = \theta_n(\eta')$ and the claim of the lemma holds. If $\eta'' > \eta'$, the fact that $(\bar{\theta}, \hat{\theta}'')$ is a local minimizer of $\Phi^{\eta''}$ implies that $\int_{\hat{\theta}''}^{\bar{\theta}} c_{q\theta}(q(\hat{\theta}'', \eta''), x)\, dx \geq 0$ and $\int_{q(\hat{\theta}'', \eta'')}^{q(\bar{\theta}, \eta'')} c_{q\theta}(y, \bar{\theta})\, dy \geq 0$. As $q(\cdot, \eta)$ increases in $\eta$ (and because of $c_{qq\theta} < 0$ and $c_{q\theta\theta} > 0$), these two inequalities hold for $\eta'$ strictly (given $\eta' < \eta''$), i.e. $\int_{\hat{\theta}''}^{\bar{\theta}} c_{q\theta}(q(\hat{\theta}'', \eta'), x)\, dx > 0$ and $\int_{q(\hat{\theta}'', \eta')}^{q(\bar{\theta}, \eta')} c_{q\theta}(y, \bar{\theta})\, dy > 0$. Hence, $(\bar{\theta}, \hat{\theta}''')$ is, for some $\hat{\theta}''' \leq \theta''$, a local minimizer of $\Phi^{\eta'}$ which means $\theta_n(\eta') = \bar{\theta}$ and $\hat{\theta}''' = \hat{\theta}_n(\eta')$.

If $\theta_n(\eta') = \bar{\theta}$ and $\eta(\bar{\theta}) > \eta'$, consider $\hat{\theta}' = sup\{\theta : \eta(\theta) = \eta'\}$. If $\hat{\theta}'$ exists, then (NLIC) has to bind to $\hat{\theta}'$ from a higher type which implies $\hat{\theta}' = \hat{\theta}_n(\eta')$ and the type from which (NLIC) binds is $\theta_n(\eta') = \bar{\theta}$. If $\hat{\theta}'$ does not exist, then $\eta(\theta) > \eta'$ for all types which implies that (NLIC) binds from $\bar{\theta}$ to $\underline{\theta}$. Hence, $\int_{\underline{\theta}}^{\bar{\theta}} c_{q\theta}(q(\underline{\theta}), x)\, dx \geq 0$ and $\int_{q(\underline{\theta})}^{q(\bar{\theta})} c_{q\theta}(y, \bar{\theta})\, dy \geq 0$. As $q(\theta) > q(\theta, \eta')$ for all types, this implies $\int_{\underline{\theta}}^{\bar{\theta}} c_{q\theta}(q(\underline{\theta}, \eta'), x)\, dx > 0$ and $\int_{q(\underline{\theta}, \eta')}^{q(\bar{\theta}, \eta')} c_{q\theta}(y, \bar{\theta})\, dy > 0$, i.e. $(\bar{\theta}, \underline{\theta})$ is a local minimizer of $\Phi^{\eta'}$ which by the structure of case 1 implies that $n = 1$ and $\theta_1(\eta') = \bar{\theta}$ and $\hat{\theta}_1(\eta') = \underline{\theta}$. Clearly, the lemma holds in this case by proposition 1.

Third, the lemma holds for $i < n$ by induction: Let the result be true for all $j > i$.

As a first case, let $\Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) < 0$. Then $\eta(\theta) \geq \eta'$ for all $\theta \in (\hat{\theta}_i(\eta'), \theta_i(\eta'))$: This was shown in the first step. Given that $\Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) < 0$ and that there is no $\Phi^{\eta'}$ minimizer within $(\hat{\theta}_i(\eta'), \theta_i(\eta'))$, it follows that $\eta(\theta) > \eta'$ for all $\theta \in (\hat{\theta}_i(\eta'), \theta_i(\eta'))$.

Define $\theta'_i = inf\{\theta : \eta(\theta) = \eta' \text{ and } \theta \geq \theta_i(\eta')\}$. Note that $\eta(\hat{\theta}_{i+1}(\eta')) = \eta'$ by the induction hypothesis and therefore $\theta'_i$ exists and $\theta'_i \leq \hat{\theta}_{i+1}$. By the definition of $\theta'_i$ and the fact that $\eta(\theta) > \eta'$ for $\theta \in (\hat{\theta}_i(\eta'), \theta_i(\eta'))$, (NLIC) binds from $\theta'_i$ (lemma 7). Hence, $\hat{\theta}_{i+1}(\eta') \neq \theta'_i$ by lemma 5 and the induction hypothesis. Theorem 2 implies then that $\theta'_i$ has to be part of a local $\Phi^{\eta'}$ minimizer which implies $\theta'_i = \theta_i(\eta')$ (because there is no

[34] If no such type exists, incentive compatibility is clearly violated as $\Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) \leq 0$ by assumption.

other $\Phi^{\eta'}$ minimizer between $\theta_i(\eta')$ and $\hat{\theta}_{i+1}(\eta')$ by the assumptions of case 1). Hence, (NLIC) binds from $\theta_i(\eta')$ to $\hat{\theta}'_i(\eta')$.

The second subcase is $\Phi^{\eta'}(\theta_i, \hat{\theta}_i) = 0$ and $\eta(\theta) < \eta'$ for some $\theta \in (\theta_i, \hat{\theta}_{i+1})$. Define $\theta'_i = inf\{\theta : \eta(\theta) < \eta' \text{ and } \theta \geq \theta_i(\eta')\}$. As $\eta(\theta_i) \geq \eta'$ by the first step, $\theta'_i \geq \theta_i$. By lemma 7, (NLIC) has to bind from $\theta'_i$ which means $\theta'_i$ has to be part of a $\Phi^{\eta'}$ minimizer by theorem 2. Therefore, $\theta'_i = \theta_i$ and again (NLIC) binds from $\theta_i(\eta')$ to $\hat{\theta}'_i(\eta')$.

It remains to show that a third case, i.e. $\Phi^{\eta'}(\theta_i, \hat{\theta}_i) = 0$ and $\eta(\theta) = \eta'$ for all $\theta \in (\theta_i, \hat{\theta}_{i+1})$ is impossible.[35] Indeed, I will show that such a decision function is not optimal.[36] Note that (NLIC) is slack for all types in $(\theta_i, \hat{\theta}_{i+1})$ in this case as $\eta$ is constant. Therefore, the following changed decision is implementable for $\varepsilon > 0$ small enough

$$q^c(\theta) = \begin{cases} (1-\varepsilon)q(\theta) + \varepsilon q(\frac{\theta_i + \hat{\theta}_{i+1}}{2}) & \text{if } \theta \in [\frac{\theta_i+\hat{\theta}_{i+1}}{2}, \frac{\theta_i + 3\hat{\theta}_{i+1}}{4}] \\ q(\theta) & \text{else.} \end{cases}$$

This decision clearly increases the principal's payoff as the changed decision is closer to $q^r$ and the principal's objective is strictly concave (lemma 2). The changed decision is implementable for $\varepsilon > 0$ small enough: This is true by the continuity of $\Phi$ in $\varepsilon$ and because $\Phi(\theta, \hat{\theta}) > 0$ for all $\theta \neq \hat{\theta}$ if $\hat{\theta} \in (\theta_i, \hat{\theta}_{i+1})$.[37] □

*Case 2a (simply nested minimizers).* Assume that the local minimizers of $\Phi^\eta$ are nested, i.e. $\hat{\theta}_i(\eta) < \hat{\theta}_{i+1}(\eta) < \theta_{i+1}(\eta) < \theta_i(\eta)$ for $i = 1, \ldots, n-1$. The following lemma states that the global minimizer of $\Phi^\eta$ is relevant in case of nested minimizers.

**Lemma WA.6.** *If $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) < \Phi^\eta(\theta_j(\eta), \hat{\theta}_j(\eta))$ for all $j = 1, \ldots, n,\ j \neq i$ and $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) \leq 0$, then (NLIC) binds from $\theta_i(\eta)$ to $\hat{\theta}_i(\eta)$ and does not bind from $\theta_j$ to $\hat{\theta}_j$ for $j \neq i$.*
*If $\Phi^\eta$ has several minimizer, i.e. $\Phi^\eta(\theta_i, \hat{\theta}_i) = \min_{\theta,\hat{\theta}} \Phi^\eta(\theta, \hat{\theta}) \leq 0$ for all $i \in I \subseteq \{1, \ldots, n\}$, then (NLIC) binds from $\theta_{\min I}(\eta)$ to $\hat{\theta}_{\min I}(\eta)$. If and only if $\min_{\theta,\hat{\theta}} \Phi^\eta(\theta, \hat{\theta}) < 0$, (NLIC) also binds from $\theta_{\max I}(\eta)$ to $\hat{\theta}_{\max I}(\eta)$. (NLIC) does not bind from $\theta_i(\eta)$ to $\hat{\theta}_i(\eta)$ if $i \neq \max I$ and $i \neq \min I$.*

It should be noted that lemma WA.5 and lemma WA.6 can be combined, i.e. if there are several separated nests then lemma WA.6 applies for each nest.

**Proof of lemma WA.6:** Take an arbitrary $\eta' \geq 0$. Theorem 2 already established that (NLIC) cannot bind from $\theta_i(\eta')$ to $\hat{\theta}_i(\eta')$ if $\Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta')) > 0$. Let $(\theta_i(\eta'), \hat{\theta}_i(\eta'))$ be a global $\Phi^{\eta'}$ minimizer and denote by $(\theta_j(\eta'), \hat{\theta}_j(\eta'))$ a local $\Phi^{\eta'}$ minimizer such that $0 \geq \Phi^{\eta'}(\theta_j(\eta'), \hat{\theta}(\eta')) > \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta'))$. For now, assume that $(\theta_i(\eta'), \hat{\theta}_i(\eta'))$ is the unique global minimizer of $\Phi^{\eta'}$. For notational simplicity, I drop the argument "$(\eta')$" in the remainder of the proof.

I start with the case where $\eta(\theta) = \eta'$ for some type $\theta$. In this case, (NLIC) has to bind for (at least) one of the $\Phi^{\eta'}$ minimizers because of theorem 2 and the preliminary result in the proof of proposition 1.

Suppose, contrary to the lemma, that (NLIC) binds from $\theta_j$ to $\hat{\theta}_j$ in the solution to problem (SB). There are two possibilities:

First, suppose $\hat{\theta}_j < \hat{\theta}_i < \theta_i < \theta_j$. But this violates theorem 2 which states that $(\theta_j, \hat{\theta}_j)$ has to minimize $\Phi^{\eta'}$ on $[\hat{\theta}_j, \theta_j]$ if (NLIC) binds from $\theta_j$ to $\hat{\theta}_j$.

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[35] Other cases cannot occur: By the induction hypothesis $\eta(\hat{\theta}_{i+1}) = \eta'$ and $\eta$ is increasing around $\hat{\theta}_{i+1}$. Therefore, $\eta(\theta) \leq \eta'$ for all $\theta \in (\theta_i, \hat{\theta}_{i+1})$: If there was a $\theta' \in (\theta_i, \hat{\theta}_{i+1})$ with $\eta(\theta') > \eta'$, there would have to be a type $\theta'' \in (\theta', \hat{\theta}_{i+1})$ with $\eta(\theta'') = \eta'$ from which (NLIC) binds (lemma 7). But by assumption there is no $\Phi^{\eta'}$ minimizer there.

[36] Here I use an argument that is based on optimality not only on incentive compatibility. However, this case is actually not a problem for the algorithm anyway: If the algorithm assigns the shadow value $\eta'$ to $\hat{\theta}_i$ and $\theta_i$, this will lead to the correct solution even if (NLIC) is not binding from $\theta_i$ to $\hat{\theta}_i$ in the strict definition I use in the paper.

[37] In fact, $min_{\hat{\theta} \in [\frac{\theta_i+\hat{\theta}_{i+1}}{2}, \frac{\theta_i+3\hat{\theta}_{i+1}}{4}],\, \theta \geq \hat{\theta}+\varepsilon'} \Phi(\theta, \hat{\theta}) > 0$. If it was not, some type in $[\frac{\theta_i+\hat{\theta}_{i+1}}{2}, \frac{\theta_i+3\hat{\theta}_{i+1}}{4}]$ would have to be a local minimizer of $\Phi^{\eta'}$ but by assumption there is no local minimizer of $\Phi^{\eta'}$ in $(\theta_i, \hat{\theta}_{i+1})$. Choosing $\varepsilon' > 0$ small enough (NLIC) can – under the changed decision – also not bind from $\theta'$ to $\hat{\theta}'$ with $\theta' - \hat{\theta}' < \varepsilon'$ because $q^c(\theta) > q^r(\theta) > s(\theta)$ for $\theta \in (\theta_i, \hat{\theta}_{i+1})$ and $\varepsilon > 0$ small enough.

Second, let $\hat{\theta}_i < \hat{\theta}_j < \theta_j < \theta_i$. Note that $\eta(\theta) \leq \eta'$ for all $\theta \in (\hat{\theta}_i, \hat{\theta}_j) \cup (\theta_j, \theta_i)$: To see this, suppose there was a type $\theta' \in (\theta_j, \theta_i)$ such that $\eta(\theta') > \eta'$ and define $\theta'' = sup\{\theta : \eta(\theta) = \eta' \text{ and } \theta < \theta'\}$. (NLIC) binds to $\theta''$ by lemma 7. As (NLIC) cannot bind from and to $\theta_j$ (see lemma 5 and the discussion below this lemma), $\theta'' > \theta_j$. By theorem 2, $\theta''$ is part of a $\Phi^{\eta'}$ minimizer, i.e. $\theta'' = \hat{\theta}_k(\eta')$ for some $k \in \{1, 2, \ldots, n\}$. By the nested structure of case 2, this is impossible as $\hat{\theta}_k(\eta') < \theta_j$ for all $k = 1, \ldots, n$. A similar argument applies for types in $(\hat{\theta}_i, \hat{\theta}_j)$. Hence, $\eta(\theta) \leq \eta'$ for all $\theta \in (\hat{\theta}_i, \hat{\theta}_j) \cup (\theta_j, \theta_i)$.

This implies $q(\theta) \leq q(\theta, \eta')$ for all $\theta \in (\hat{\theta}_i, \hat{\theta}_j) \cup (\theta_j, \theta_i)$ and therefore

$$\Phi(\theta_i, \hat{\theta}_i) = \Phi(\theta_j, \hat{\theta}_j) + \Phi^{\eta'}(\theta_i, \hat{\theta}_i) - \Phi^{\eta'}(\theta_j, \hat{\theta}_j) + \int_{\hat{\theta}_i}^{\hat{\theta}_j} \int_{q(x)}^{q(x,\eta')} c_{q\theta}(y, x)\, dy\, dx$$
$$+ \int_{\theta_j}^{\theta_i} \int_{q(x)}^{q(x,\eta')} c_{q\theta}(y, x)\, dy\, dx \leq \Phi^{\eta'}(\theta_i, \hat{\theta}_i) - \Phi^{\eta'}(\theta_j, \hat{\theta}_j) < 0.$$

This contradicts incentive compatibility of the optimal decision function.

Hence, (NLIC) binds for a global $\Phi^{\eta'}$ minimizer if $\eta(\theta) = \eta'$ for some type.

If $\eta(\theta) < \eta'$ for all types then $q(\theta) < q(\theta, \eta')$ for all types and therefore $\Phi(\theta_i, \hat{\theta}_i) = \Phi^{\eta'}(\theta_i, \hat{\theta}_i) + \int_{\hat{\theta}_i}^{\theta_i} \int_{q(\hat{\theta}_i)}^{q(\hat{\theta}_i,\eta')} -c_{q\theta}(y, x)\, dy\, dx - \int_{\hat{\theta}_i}^{\theta_i} \int_{q(x)}^{q(x,\eta')} -c_{q\theta}(y, x)\, dy\, dx < \Phi^{\eta'}(\theta_i, \hat{\theta}_i) \leq 0$ where the first inequality follows from $c_{qq\theta} < 0$ and $\int_{\hat{\theta}_i}^{\theta_i} c_{q\theta}(q(\hat{\theta}_i, \eta'), x)\, dx \geq 0$ because $(\theta_i, \hat{\theta}_i)$ locally minimize $\Phi^{\eta'}$. This violates incentive compatibility. Hence, $\eta(\theta) < \eta'$ for all types is impossible.

If $\eta(\theta) > \eta' \geq 0$ for all types then (NLIC) binds from $\bar{\theta}$ to $\underline{\theta}$. Hence, $\int_{\underline{\theta}}^{\bar{\theta}} c_{q\theta}(q(\underline{\theta}), x)\, dx \geq 0$ and $\int_{q(\underline{\theta})}^{q(\bar{\theta})} c_{q\theta}(y, \bar{\theta})\, dx \geq 0$. This implies by $q(\theta) > q(\theta, \eta')$ that $\int_{\underline{\theta}}^{\bar{\theta}} c_{q\theta}(q(\underline{\theta}, \eta'), x)\, dx > 0$ and $\int_{q(\underline{\theta},\eta')}^{q(\bar{\theta},\eta')} c_{q\theta}(y, \bar{\theta})\, dx > 0$. Hence, $\hat{\theta}_1(\eta') = \underline{\theta}$ and $\theta_1(\eta') = \bar{\theta}$ and $(\bar{\theta}, \underline{\theta})$ is a local $\Phi^{\eta'}$ minimizer. If $(\bar{\theta}, \underline{\theta})$ was not the unique global $\Phi^{\eta'}$ minimizer, the same argument as in the first step would establish that there is a $\theta_i$ and $\hat{\theta}'_i$ with $\Phi(\theta'_i, \hat{\theta}'_i) < 0$. Hence, $(\bar{\theta}, \underline{\theta})$ is the unique global $\Phi^{\eta'}$ minimizer in this case and the lemma holds.

It remains to analyze the case where there are several global $\Phi^{\eta'}$ minimizers. In this case, (NLIC) will bind for the most exterior and most interior global $\Phi^{\eta'}$ minimizer, i.e. $\theta_i$ to $\hat{\theta}_i$ for the highest and lowest $i$ among the global minimizers. To see this, let $\theta_j = max\{\theta : \eta(\theta) = \eta'\}$. Then (NLIC) binds from $\theta_j$ to $\hat{\theta}_j = min\{\theta : \eta(\theta) = \eta'\}$ and $\eta(\theta) < \eta'$ for all $\theta \in [\underline{\theta}, \hat{\theta}_j) \cup (\theta_j, \bar{\theta}]$. Suppose there existed a global $\Phi^{\eta'}$ minimizer $(\theta_i, \hat{\theta}_i)$ such that $\hat{\theta}_i < \hat{\theta}_j < \theta_j < \theta_i$. As $q(\theta) < q(\theta, \eta')$ for $\theta \in [\underline{\theta}, \hat{\theta}_j) \cup (\theta_j, \bar{\theta}]$, $\Phi(\theta_i, \hat{\theta}_i) < \Phi(\theta_j, \hat{\theta}_j) + \Phi^{\eta'}(\theta_i, \hat{\theta}_i) - \Phi^{\eta'}(\theta_j, \hat{\theta}_j) \leq 0$ contradicting incentive compatibility. Hence, there is no global $\Phi^{\eta'}$ minimizer $(\theta_i, \hat{\theta}_i)$ such that $\hat{\theta}_i < \hat{\theta}_j < \theta_j < \theta_i$, i.e. (NLIC) binds for the most exterior global $\Phi^{\eta'}$ minimizer.

To show the same for the most interior, let $\theta_j$ be the lowest type such that (i) $\eta(\theta) = \eta'$ and (ii) (NLIC) binds from $\theta_j$. Assume $\Phi^{\eta'}(\theta_j, \hat{\theta}_j) < 0$ which implies that $\eta(\theta) > \eta'$ for all $\theta \in (\hat{\theta}_j, \theta_j)$ as everything else contradicts either the definition of $\theta_j$ or incentive compatibility between $\theta_j$ and $\hat{\theta}_j$. Suppose there was a global $\Phi^{\eta'}$ minimizer $(\theta_i, \hat{\theta}_i)$ such that $\hat{\theta}_j < \hat{\theta}_i < \theta_i < \theta_j$. Define $\hat{\theta}'$ by $q(\hat{\theta}') = q(\hat{\theta}_i, \eta')$. Then, because $q(\theta) > q(\theta, \eta')$ for all $\theta \in (\hat{\theta}_j, \hat{\theta}_i) \cup (\theta_i, \theta_j)$, $\Phi(\theta_i, \hat{\theta}') < \Phi(\theta_j, \hat{\theta}_j) - \Phi^{\eta'}(\theta_j, \hat{\theta}_j) + \Phi^{\eta'}(\theta_i, \hat{\theta}_i) = 0$ which contradicts incentive compatibility. Hence, there cannot be such an $(\hat{\theta}_i, \theta_i)$ which means (NLIC) binds for the most interior global $\Phi^{\eta'}$ minimizer.

If the most interior global $\Phi^{\eta'}$ minimizer is denoted by $(\theta_i, \hat{\theta}_i)$ and the most exterior $\Phi^{\eta'}$ minimizer is denoted by $(\theta_e, \hat{\theta}_e)$, then $\eta(\theta') = \eta'$ for all $\theta \in (\hat{\theta}_e, \hat{\theta}_i) \cup (\theta_i, \theta_e)$: As $0 = \Phi(\theta_e, \hat{\theta}_e) - \Phi(\theta_i, \hat{\theta}_i) = \Phi^{\eta'}(\theta_e, \hat{\theta}_e) - \Phi^{\eta'}(\theta_i, \hat{\theta}_i) + \int_{\hat{\theta}_e}^{\hat{\theta}_i} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx + \int_{\theta_i}^{\theta_e} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx = \int_{\hat{\theta}_e}^{\hat{\theta}_i} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx + \int_{\theta_i}^{\theta_e} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx$, $\eta$ cannot be above (or below) $\eta'$ for all types in $(\theta_i, \theta_e)$ (because of (C3) this would imply that $\eta$ is above (below) $\eta'$ for all types in $(\hat{\theta}_e, \hat{\theta}_i)$). If there was a type $\theta' \in (\theta_i, \theta_e)$ such that $\eta(\theta') > \eta'$, then (NLIC) would have to bind to $\hat{\theta}' = sup\{\theta : \eta(\theta) = \eta' \text{ and } \theta < \theta'\}$. As $\hat{\theta}' \geq \theta_i$, this is impossible as there is no $\Phi^{\eta'}$ minimizer $(\theta_k, \hat{\theta}_k)$ with $\hat{\theta}_k \geq \theta_i$ by the assumption of case 2a. □

*Case 2b (multiply nested minimizers).* Assume that $\hat{\theta}_1(\eta) < \hat{\theta}_2(\eta) < \theta_2(\eta) < \cdots < \hat{\theta}_n(\eta) < \theta_n(\eta) < \theta_1(\eta)$, i.e. several separated $\Phi^\eta$ minimizers are nested within another minimizer. By theorem 2, (NLIC) can only bind from $\theta_i(\eta)$ to $\hat{\theta}_i(\eta)$ if $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) \leq 0$. A $\Phi^\eta$ minimizer not satisfying this inequality is irrelevant for determining the solution. To avoid cluttering notation by case distinction I will therefore not consider such minimizers and assume $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) \leq 0$ for $i = 1, \ldots, n$.

**Lemma WA.7.** *Given the assumptions of case 2b, (NLIC) binds from $\theta_1$ to $\hat{\theta}_1$ if and only if $\Phi^\eta(\theta_1(\eta), \hat{\theta}_1(\eta)) \leq \sum_{i=2}^n \Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta))$. For $i \geq 2$, (NLIC) binds from $\theta_i$ to $\hat{\theta}_i$ if and only if $\Phi^\eta(\theta_1(\eta), \hat{\theta}_1(\eta)) \geq \sum_{i=2}^n \Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta))$ and $\Phi^\eta(\theta_i(\eta), \hat{\theta}_i(\eta)) \leq 0$ and one of these two inequalities holds strictly.*

**Proof of lemma WA.7:** Take an $\eta' \geq 0$ at which the structure of local $\Phi^{\eta'}$ minimizers is as in case 2b. The proof that (NLIC) binds from some $\theta_i(\eta')$ to some $\hat{\theta}_i(\eta')$ if $\Phi^{\eta'}(\theta_j(\eta'), \hat{\theta}_j(\eta')) \leq 0$ for some $j = 1, \ldots, n$ is analogous to lemma WA.6 and therefore not duplicated.

First, let $\Phi^{\eta'}(\theta_1(\eta'), \hat{\theta}_1(\eta')) < \sum_{i=2}^n \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta'))$. Suppose (NLIC) was binding from $\theta_j(\eta')$ to $\hat{\theta}_j(\eta')$ for all $j \in J \subseteq \{2, \ldots, n\}$ (and possibly also from $\theta_1(\eta')$ to $\hat{\theta}_1(\eta')$). To shorten notation, I will omit the argument "$(\eta')$" for the remainder of the proof.

Let $j$ and $k$ be consecutive elements of $J$, i.e. $j, k \in J$, $j < k$ and $\nexists z \in J. j < z < k$. Then, $\eta(\theta) = \eta'$ for all $\theta \in [\theta_j, \hat{\theta}_k]$ and $\eta(\theta) \leq \eta'$ for $\theta \in [\hat{\theta}_1, \hat{\theta}_{\min J}] \cup [\theta_{\max J}, \theta_1]$. To show this, suppose there was a $\theta' \in [\theta_j, \hat{\theta}_k]$ with $\eta(\theta') > \eta'$. Define $\hat{\theta}' = sup\{\theta : \eta(\theta) = \eta' \text{ and } \theta < \theta'\}$. (NLIC) has to bind to $\hat{\theta}'$ (lemma 7). Note that $\hat{\theta}' \geq \theta_j$ as $\eta(\theta_j) = \eta'$ because (NLIC) binds from $\theta_j$. Actually, $\hat{\theta}' > \theta_j$ by lemma 5. But then, by theorem 2, $\hat{\theta}'$ has to be part of a $\Phi^{\eta'}$ minimizer although there is none (to which (NLIC) binds) in $(\theta_j, \hat{\theta}_k)$ by assumption. Hence, $\eta(\theta) = \eta'$ for all $\theta \in (\theta_j, \hat{\theta}_k)$ which implies $\eta(\hat{\theta}_k) = \eta'$. This implies $\eta(\theta_k) = \eta'$: If (NLIC) binds from $\theta_k$ to $\hat{\theta}_k$ this is implied by (C3). If (NLIC) does not bind, then $\eta(\theta) = \eta'$ for all $\theta \in (\hat{\theta}_k, \theta_k)$ as there is not $\Phi^{\eta'}$ minimizer in $(\hat{\theta}_k, \theta_k)$ by assumption. Given that $\eta(\theta_k) = \eta(\hat{\theta}_k) = \eta'$, one can iterate the previous argument for $k + 1$. All other cases are dealt with similarly.

The last paragraph implies that $q(\theta) \leq q(\theta, \eta')$ for $\theta \in \bigcup_{j \in J \setminus \{\max J\}} [\theta_j, \hat{\theta}_k] \cup [\hat{\theta}_1, \hat{\theta}_{\min J}] \cup [\theta_{\max J}, \theta_1]$. Therefore,

$$
\begin{aligned}
\Phi(\theta_1, \hat{\theta}_1) &\leq \Phi^{\eta'}(\theta_1, \hat{\theta}_1) - \left( \sum_{j \in J} \Phi^{\eta'}(\theta_j, \hat{\theta}_j) \right) + \sum_{j \in J} \Phi(\theta_j, \hat{\theta}_j) \\
&= \Phi^{\eta'}(\theta_1, \hat{\theta}_1) - \sum_{j \in J} \Phi^{\eta'}(\theta_j, \hat{\theta}_j) < 0
\end{aligned}
\tag{WA.5}
$$

where the first inequality holds strictly if $\eta(\theta) < \eta'$ for some $\theta \in [\hat{\theta}_1, \hat{\theta}_{\min J}] \cup [\theta_{\max J}, \theta_1]$ and the second inequality holds strictly by $\sum_{j=2}^k \Phi^{\eta'}(\theta_j, \hat{\theta}_j) > \Phi^{\eta'}(\theta_1, \hat{\theta}_1)$. Hence, incentive compatibility of the optimal decision is contradicted. Therefore, (NLIC) cannot bind from $\theta_j(\eta')$ to $\hat{\theta}_j(\eta')$ for $j = 2, \ldots, k \leq n$ if $\Phi^{\eta'}(\theta_1(\eta'), \hat{\theta}_1(\eta')) < \sum_{i=2}^n \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta'))$. (NLIC) has then to bind from $\theta_1$ to $\hat{\theta}_1$ only – as stated in the lemma.

Second, let $\sum_{j=2}^k \Phi^{\eta'}(\theta_j, \hat{\theta}_j) = \Phi^{\eta'}(\theta_1, \hat{\theta}_1)$. The same steps as above still hold but WA.5 holds with equality if $J = \{1, \ldots, n\}$. Hence, (NLIC) binds still from $\theta_1$ to $\hat{\theta}_1$ as $\eta < \eta'$ for $\theta > \theta_1$ (and for $\theta < \hat{\theta}_1$): Otherwise, another $\Phi^{\eta'}$ minimizer $(\theta_0, \hat{\theta}_0)$ with $\theta_1 < \theta_0$ (or $\hat{\theta}_0 < \hat{\theta}_1$) would have to exist which was ruled out by assumption in case 2b.

Recall that (NLIC) binding from $\theta_1$ to $\hat{\theta}_1$ implies $\eta(\theta) \geq \eta'$ for all $\theta \in (\hat{\theta}_1, \theta_1)$ (lemma 9). Therefore, (NLIC) does not bind from $\theta_j$ to $\hat{\theta}_j$ if $\Phi^{\eta'}(\theta_j, \hat{\theta}_j) = 0$; i.e. if $\Phi(\theta_j, \hat{\theta}_j) = 0$, then $\eta$ is constant in neighborhoods of $\theta_j$ and $\hat{\theta}_j$.

However, (NLIC) binds from $\theta_i$ to $\hat{\theta}_i$ if $\Phi^{\eta'}(\theta_i, \hat{\theta}_i) < 0$: Suppose not. For incentive compatibility, $\eta(\theta) > \eta'$ for some $\theta \in (\hat{\theta}_i, \theta_i)$ and – using the same arguments as above – this implies $\eta(\theta) > \eta'$ for all $\theta \in (\hat{\theta}_i, \theta_i)$. If (NLIC) does not bind from $\theta_i$ to $\hat{\theta}_i$, this conclusion even holds true at $\theta_i$ and $\hat{\theta}_i$, i.e. $\eta(\theta) > \eta'$ for all

$\theta \in [\hat{\theta}_i, \theta_i]$. Then,

$$\begin{aligned}
\Phi(\theta_1, \hat{\theta}_1) &= \Phi^{\eta'}(\theta_1, \hat{\theta}_1) - \left(\sum_{j=2}^{n} \Phi^{\eta'}(\theta_j, \hat{\theta}_j)\right) + \sum_{j=2}^{n} \left(\Phi(\theta_j, \hat{\theta}_j) + \int_{\hat{\theta}_j}^{\theta_j} \int_{q(\hat{\theta}_j, \eta')}^{q(\hat{\theta}_j)} -c_{q\theta}(y, x)\, dy\, dx\right) \\
&\quad + \int_{\hat{\theta}_1}^{\hat{\theta}_2} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx + \int_{\theta_n}^{\theta_1} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx \\
&\quad + \sum_{j=2}^{n-1} \int_{\theta_j}^{\hat{\theta}_{j+1}} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx \\
&\geq \sum_{j=2}^{n} \left(\Phi(\theta_j, \hat{\theta}_j) + \int_{\hat{\theta}_j}^{\theta_j} \int_{q(\hat{\theta}_j, \eta')}^{q(\hat{\theta}_j)} -c_{q\theta}(y, x)\, dy\, dx\right) \\
&\geq \Phi(\theta_i, \hat{\theta}_i) + \sum_{j=2}^{n} \left(\int_{\hat{\theta}_j}^{\theta_j} \int_{q(\hat{\theta}_j, \eta')}^{q(\hat{\theta}_j)} -c_{q\theta}(y, x)\, dy\, dx\right) > 0
\end{aligned}$$

where the first inequality follows from $q(x) \geq q(x, \eta') > s(x)$ and the assumption $\Phi^{\eta'}(\theta_1(\eta'), \hat{\theta}_1(\eta')) = \sum_{i=2}^{n} \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta'))$. The second inequality follows from incentive compatibility. The third inequality holds as $(\theta_j, \hat{\theta}_j)$ is a local minimizer of $\Phi^{\eta'}$ and therefore the first-order condition $\int_{\hat{\theta}_j}^{\theta_j} -c_{q\theta}(q(\hat{\theta}_j), x)\, dx = 0$ has to hold. Together with $c_{qq\theta} < 0$, this implies the third inequality which is strict as $q(\hat{\theta}_i) > q(\hat{\theta}_i, \eta')$ by $\eta(\hat{\theta}_i) > \eta'$. This contradicts that (NLIC) binds from $\theta_1$ to $\hat{\theta}_1$. Hence, (NLIC) has to bind from $\theta_i$ to $\hat{\theta}_i$.

Third, let $\Phi^{\eta'}(\theta_1(\eta'), \hat{\theta}_1(\eta')) > \sum_{i=2}^{n} \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta'))$. Suppose, contrary to the lemma, that (NLIC) was binding from $\theta_1$ to $\hat{\theta}_1$. Then $\eta(\theta) \geq \eta'$ for all $\theta \in (\hat{\theta}_1, \theta_1)$ by lemma 9 which implies $q(\theta) \geq q(\theta, \eta')$ for these types. Then,

$$\begin{aligned}
\Phi(\theta_1, \hat{\theta}_1) &= \Phi^{\eta'}(\theta_1, \hat{\theta}_1) - \left(\sum_{j=2}^{n} \Phi^{\eta'}(\theta_j, \hat{\theta}_j)\right) + \sum_{j=2}^{n} \left(\Phi(\theta_j, \hat{\theta}_j) + \int_{\hat{\theta}_j}^{\theta_j} \int_{q(\hat{\theta}_j, \eta')}^{q(\hat{\theta}_j)} -c_{q\theta}(y, x)\, dy\, dx\right) \\
&\quad + \int_{\hat{\theta}_1}^{\hat{\theta}_2} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx + \int_{\theta_n}^{\theta_1} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx \\
&\quad + \sum_{j=2}^{n-1} \int_{\theta_j}^{\hat{\theta}_{j+1}} \int_{q(x,\eta')}^{q(x)} -c_{q\theta}(y, x)\, dy\, dx \\
&> \sum_{j=2}^{n} \left(\Phi(\theta_j, \hat{\theta}_j) + \int_{\hat{\theta}_j}^{\theta_j} \int_{q(\hat{\theta}_j, \eta')}^{q(\hat{\theta}_j)} -c_{q\theta}(y, x)\, dy\, dx\right) \\
&\geq \sum_{j=2}^{n} \int_{\hat{\theta}_j}^{\theta_j} \int_{q(\hat{\theta}_j, \eta')}^{q(\hat{\theta}_j)} -c_{q\theta}(y, x)\, dy\, dx > 0
\end{aligned}$$

where the first inequality follows from $q(x) \geq q(x, \eta') > s(x)$ and the assumption $\Phi^{\eta'}(\theta_1(\eta'), \hat{\theta}_1(\eta')) > \sum_{i=2}^{n} \Phi^{\eta'}(\theta_i(\eta'), \hat{\theta}_i(\eta'))$. The second inequality follows from incentive compatibility. The third inequality holds as $(\theta_j, \hat{\theta}_j)$ is a local minimizer of $\Phi^{\eta'}$ and therefore the first-order condition $\int_{\hat{\theta}_j}^{\theta_j} -c_{q\theta}(q(\hat{\theta}_j), x)\, dx = 0$ has to hold. Together with $c_{qq\theta} < 0$, this implies the third inequality. This contradicts that (NLIC) binds from $\theta_1$ to $\hat{\theta}_1$. Having ruled this possibility out, lemma WA.5 implies the result. □

*Case 3 (overlapping minimizers).* Assume that some local $\Phi^{\eta}$ minimizers overlap, e.g. $\hat{\theta}_1(\eta) < \hat{\theta}_2(\eta) < \theta_1(\eta) < \theta_2(\eta)$. It turns out that this situation is equivalent to a multiply nested situation as in case 2b.

Theorem 2 implies that (NLIC) cannot bind from $\theta_i(\eta)$ to $\hat{\theta}_i(\eta)$ if those two types do not minimize $\Phi^\eta$ on $[\hat{\theta}_i(\eta), \theta_i(\eta)]$. Hence, such $\Phi^\eta$ are not relevant and not taken into consideration from the outset. Using a similar argument as in the proof of lemma 5, the following result is obtained:

**Lemma WA.8.** *For $i = 1, 2$, let $(\theta_i, \hat{\theta}_i)$ minimize $\Phi^\eta$ locally and let $(\theta_i, \hat{\theta}_i)$ minimize $\Phi^\eta$ on $[\hat{\theta}_i, \theta_i]$. Assume $\hat{\theta}_1 < \hat{\theta}_2 < \theta_1 < \theta_2$. Then there exists a $(\theta_0, \hat{\theta}_0)$ such that (i) $(\theta_0, \hat{\theta}_0)$ minimizes $\Phi^\eta$ locally, (ii) $\hat{\theta}_0 \leq \hat{\theta}_1$ and $\theta_0 \geq \theta_2$, (iii) $\Phi^\eta(\theta_0, \hat{\theta}_0) < min\{\Phi^\eta(\theta_1, \hat{\theta}_1), \Phi^\eta(\theta_2, \hat{\theta}_2)\}$.*

In words, if two $\Phi^\eta$ minimizers overlap, there exists a nesting $\Phi^\eta$ minimizer. Recall lemma 5 which implies that (NLIC) cannot bind both from $\theta_1$ to $\hat{\theta}_1$ and from $\theta_2$ to $\hat{\theta}_2$. If there are no other $\Phi^\eta$ minimizers in $(\hat{\theta}_0, \theta_0)$, the same arguments as in lemma WA.6 imply that (NLIC) binds from $\theta_0$ to $\hat{\theta}_0$ and (NLIC) is slack between $\theta_i$ and $\hat{\theta}_i$, $i = 1, 2$, because $\Phi^\eta(\theta_0, \hat{\theta}_0) < min\{\Phi^\eta(\theta_1, \hat{\theta}_1), \Phi^\eta(\theta_2, \hat{\theta}_2)\}$.

With multiple overlapping $\Phi^\eta$ minimizers, i.e. $(\theta_i, \hat{\theta}_i)$ for $i = 1, \ldots, n$ such that $\hat{\theta}_i < \hat{\theta}_{i+1} < \theta_i < \theta_{i+1}$ for $i =, 1 \ldots, n-1$, the arguments of lemma WA.8 still imply that there is a nesting $\Phi^\eta$ minimizer $(\theta_0, \hat{\theta}_0)$ such that $\hat{\theta}_0 \leq \hat{\theta}_1$, $\theta_0 \geq \theta_n$ and $\Phi^\eta(\theta_0, \hat{\theta}_0) < \min_i \Phi^\eta(\theta_i, \hat{\theta}_i)$. However, this does not immediately imply that (NLIC) binds from $\theta_0$ to $\hat{\theta}_0$: If there is a set $J \subset \{1, 2, \ldots, n\}$ such that $\theta_j < \hat{\theta}_{j+1}$ for every $j \in J$ and $\sum_{j \in J} \Phi^\eta(\theta_j, \hat{\theta}_j) < \Phi^\eta(\theta_0, \hat{\theta}_0)$, then – following the arguments in the proof of lemma WA.7 – (NLIC) will bind from $\theta_j$ to $\hat{\theta}_j$ for all $j \in J$ and (NLIC) will be slack between $\theta_0$ and $\hat{\theta}_0$. If no such set $J$ exists, (NLIC) binds from $\theta_0$ to $\hat{\theta}_0$ and is slack between all $\theta_i$ and $\hat{\theta}_i$ within $(\hat{\theta}_0, \theta_0)$.

**Proof of lemma WA.8:** First, note that $\Phi^\eta(\theta_1, \hat{\theta}_2) \geq max\{\Phi^\eta(\theta_1, \hat{\theta}_1), \Phi^\eta(\theta_2, \hat{\theta}_2)\}$: If this was not the case, e.g. say $\Phi^\eta(\theta_1, \hat{\theta}_2) < \Phi^\eta(\theta_2, \hat{\theta}_2)$, then $(\theta_2, \hat{\theta}_2)$ would not minimize $\Phi^\eta$ on $[\hat{\theta}_2, \theta_2]$ as assumed.

Second, similarly to the proof of lemma 5

$$\begin{aligned}
\Phi^\eta(\theta_2, \hat{\theta}_1) &= \Phi^\eta(\theta_1, \hat{\theta}_1) + \Phi^\eta(\theta_2, \hat{\theta}_2) - \Phi^\eta(\theta_1, \hat{\theta}_2) + \int_{\theta_1}^{\theta_2} \int_{q(\hat{\theta}_1, \eta)}^{q(\hat{\theta}_2, \eta)} -c_{q\theta}(y, x)\, dy\, dx \\
&< \Phi^\eta(\theta_1, \hat{\theta}_1) + \Phi^\eta(\theta_2, \hat{\theta}_2) - \Phi^\eta(\theta_1, \hat{\theta}_2) \leq min\{\Phi^\eta(\theta_1, \hat{\theta}_1), \Phi^\eta(\theta_2, \hat{\theta}_2)\}
\end{aligned}$$

where the first inequality uses $\int_{q(\hat{\theta}_1, \eta}^{q(\theta_1, \eta)} c_{q\theta}(y, \theta_1)\, dy = 0$ (because $(\theta_1, \hat{\theta}_1)$ is a local $\Phi^\eta$ minimizer), $q(\hat{\theta}_2, \eta) < q(\theta_1, \eta)$ and $c_{q\theta\theta} > 0$. The second inequality follows directly from the first paragraph of this proof.

As $(\theta_1, \hat{\theta}_1)$ and $(\theta_2, \hat{\theta}_2)$ locally minimize $\Phi^\eta$, the first-order conditions $\int_{q(\hat{\theta}_2, \eta)}^{q(\theta_2, \eta)} c_{q\theta}(y, \theta_2)\, dy \geq 0$ and $\int_{\hat{\theta}_1}^{\theta_1} c_{q\theta}(q(\hat{\theta}_1, \eta), x)\, dx \geq 0$ hold. Consequently, $\int_{q(\hat{\theta}_1, \eta)}^{q(\theta_2, \eta)} c_{q\theta}(y, \theta_2)\, dy > 0$ and $\int_{\hat{\theta}_1}^{\theta_2} c_{q\theta}(q(\hat{\theta}_1, \eta), x)\, dx > 0$. Hence, $\Phi^\eta(\theta, \hat{\theta})$ is decreasing in the first and increasing in the second argument at $(\theta_2, \hat{\theta}_1)$ which implies that there has to exist a local minimizer $(\theta_0, \hat{\theta}_0)$ with the properties of the lemma. □

*Other cases.* If the local minimizers of $\Phi^\eta$ have a different structure than in the cases 1 to 3, the previous lemmas can often still be used to determine the relevant minimizers. This is true because the proofs are based on incentive compatibility. To give an example, suppose the minimizers of $\Phi^\eta$ were ordered $\hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3 < \theta_3 < \theta_2 < \hat{\theta}_4 < \theta_4 < \theta_1$. The minimizers $(\hat{\theta}_2, \theta_2)$ and $(\hat{\theta}_3, \theta_3)$ are nested. Assume $\Phi^\eta(\theta_2, \hat{\theta}_2) < \Phi^\eta(\theta_3, \hat{\theta}_3)$. In a nutshell, the proof of lemma WA.6 establishes that (NLIC) from $\theta_2$ to $\hat{\theta}_2$ would be violated if (NLIC) was binding from $\theta_3$ to $\hat{\theta}_3$. This holds still true. Therefore (NLIC) cannot bind from $\theta_3$ to $\hat{\theta}_3$ and $(\theta_3, \hat{\theta}_3)$ is an irrelevant minimizer. This leaves us with the multiply nested structure $\hat{\theta}_1 < \hat{\theta}_2 < \theta_2 < \hat{\theta}_4 < \theta_4 < \theta_1$ for which lemma WA.7 establishes the relevant minimizer.

*WA 2.6. Variational condition*

This subsection derives condition (C3) for strictly monotone and continuous solutions. First, I will present the derivation of Araujo and Moreira [5]. Note that (C1) does not play a role if only U-shaped decisions are analyzed. However, the result for monotone decisions (where (C1) is relevant) is already derived in the working paper version [33] and this is the proof that will be sketched here. Second, I will show how (C3) can also be derived using a proof technique similar to the one used to proof theorem 1.

*Sketch of proof in Araujo and Moreira [5]*

Starting from (C2), Araujo and Moreira [5] derive the following condition (with $q = q(\theta) = q(\hat{\theta})$):

$$\frac{u_q(q,\theta) - c_q(q,\theta) + \frac{1-F(\theta)}{f(\theta)} c_{q\theta}(q,\theta)}{c_{q\theta}(q,\theta)} f(\theta) = \frac{u_q(q,\hat{\theta}) - c_q(q,\hat{\theta}) + \frac{1-F(\hat{\theta})}{f(\hat{\theta})} c_{q\theta}(q,\hat{\theta})}{c_{q\theta}(q,\hat{\theta})} f(\hat{\theta}). \tag{WA.6}$$

To derive a similar condition for $q(\theta) \neq q(\hat{\theta})$, let (NLIC) bind from $\theta$ to $\hat{\theta}$ with $\theta, \hat{\theta} \in (\underline{\theta}, \bar{\theta})$ and assume that $q(\cdot)$ is strictly monotone in neigborhoods of $\theta$ and $\hat{\theta}$ and continuous at $\theta$ and $\hat{\theta}$.

Given $\theta$ and $q(\theta)$, the equation $c_\theta(q(\theta),\theta) = c_\theta(q(\hat{\theta}),\theta)$ pins down a decision $q(\hat{\theta})$ where incentive compatibility could be binding. Given this, $q(\hat{\theta})$ as well as $\theta$ and $q(\theta)$, the equation $c_q(q(\hat{\theta}),\hat{\theta}) = c_q(q(\hat{\theta}),\theta)$ determines $\hat{\theta}$. Therefore, the critical $\hat{\theta}$ can be written as a function of $\theta$ and $q(\theta)$, i.e. $\hat{\theta} = \phi(\theta, q(\theta))$.

Differentiating the two conditions, the partial derivatives $\phi_\theta$ and $\phi_q$ can be obtained as

$$\begin{aligned}
\phi_\theta(\theta,q) &= \frac{c_{q\theta}(\hat{q},\theta)}{c_{q\theta}(\hat{q},\hat{\theta})} + \frac{(c_{qq}(\hat{q},\theta) - c_{qq}(\hat{q},\hat{\theta}))(c_{\theta\theta}(q,\theta) - c_{\theta\theta}(\hat{q},\theta))}{c_{q\theta}(\hat{q},\hat{\theta})c_{q\theta}(\hat{q},\theta)}, \\
\phi_q(\theta,q) &= \frac{c_{q\theta}(q,\theta)[c_{qq}(\hat{q},\theta) - c_{qq}(\hat{q},\hat{\theta})]}{c_{q\theta}(\hat{q},\hat{\theta})c_{q\theta}(\hat{q},\theta)}
\end{aligned}$$

where $\hat{q} = q(\hat{\theta})$ and $q = q(\theta)$.

Denote by $h$ an implementable perturbation of the optimal solution $q^*$ on some interval $[\theta_1, \theta_2]$ with $h(\theta_1) = h(\theta_2) = 0$. I denote in the remainder of this section the optimal solution by $q^*(\theta)$ and the perturbed solution by $q(\theta) = q^*(\theta) + \varepsilon h(\theta)$. I will call a perturbation $h$ implementable if there exists an $\varepsilon' > 0$ such that the decision $q^* + \varepsilon h$ is implementable for all $\varepsilon \in [0, \varepsilon']$. Implementability implies that $\hat{\theta} = \phi(\theta, q(\theta))$ if the incentive constraint binds from $\theta$ to $\hat{\theta}$ (assuming strict monotonicity and continuity of $q$ for $\varepsilon \in [0, \varepsilon']$).

The idea of the variational argument is the following: I want to derive a necessary condition for a type $\theta$ such that $\Phi(\theta, \hat{\theta}) = 0$ for some $\hat{\theta}$. To do so, it is assumed that also under the perturbed decision the incentive constraint is binding for $\theta$ and some (other) $\hat{\theta}$. The type $\hat{\theta}$ to which the non-local incentive constraint binds depends on the perturbation and is given by $\phi(\theta, q(\theta))$. The way one should think about it is that incentive compatibility is binding from each $\theta \in (\theta_1, \theta_2)$ to some $\hat{\theta}$ in some interval $(\hat{\theta}_1, \hat{\theta}_2)$. The specific type $\hat{\theta}$ to which a non-local incentive constraint binds from a given $\theta$ depends on the perturbation $h$.

The part of the principal's objective function affected by the perturbation can be written as[38]

$$\begin{aligned}
G(\varepsilon) &= \int_{\theta_1}^{\theta_2} g(q(\theta),\theta)\, d\theta + \int_{\phi(\theta_2,q(\theta_2))}^{\phi(\theta_1,q(\theta_1))} g(q(\theta),\theta)\, d\theta \\
&= \int_{\theta_1}^{\theta_2} \left\{ g(q(\theta),\theta) - g(\hat{q}(\theta,q(\theta)),\phi(\theta,q(\theta)))\, [\phi_q(q(\theta),\theta)q_\theta(\theta) + \phi_\theta(q(\theta),\theta)] \right\} d\theta
\end{aligned} \tag{WA.7}$$

where $g(q(\theta),\theta) = \left[u(q(\theta),\theta) - c(q(\theta),\theta) + \frac{1-F(\theta)}{f(\theta)} c_\theta(q(\theta),\theta)\right] f(\theta)$ is the virtual valuation weighted by the density. The second line is a normal change of variables where $\hat{q}(\theta, q)$ denotes the $\hat{q}$ solving $c_\theta(q,\theta) = c_\theta(\hat{q},\theta)$ with $q \neq \hat{q}$. Note that $\partial\hat{q}/\partial q = c_{q\theta}(q,\theta)/c_{q\theta}(\hat{q},\theta)$.

Differentiating (WA.7) gives

$$G'(0) = \int_{\theta_1}^{\theta_2} \left\{ g_q h - \hat{g}((\phi_{qq} q^*_\theta + \phi_{q\theta})h + \phi_q h_\theta) - (\hat{g}_q \hat{q}_q + \hat{g}_\theta \phi_q)(\phi_q q^*_\theta + \phi_\theta)h \right\} d\theta = 0$$

[38]It follows from lemma 5 that $\phi(\theta_1, q(\theta_1)) > \phi(\theta_2, q(\theta_2))$.

where arguments are omitted and a hat denotes evaluation at $(\hat{\theta}, q^*(\hat{\theta}))$ and $\hat{\theta} = \phi(\theta, q^*(\theta))$. Integrating $\int_{\theta_1}^{\theta_2} (\hat{g}\phi_q) h_\theta \, d\theta$ by parts and substituting yields for the previous equation

$$\int_{\theta_1}^{\theta_2} \{g_q - \hat{g}_q \hat{q}_q \phi_\theta + \hat{g}_q \hat{q}_\theta \phi_q\} \, h \, d\theta = \int_{\theta_1}^{\theta_2} \left\{ g_q - \hat{g}_q \frac{c_{q\theta}(q(\theta), \theta)}{c_{q\theta}(q(\hat{\theta}), \hat{\theta})} \right\} h \, d\theta = 0.$$

I will argue that the last equation can only hold if

$$g_q(q(\theta), \theta) - g_q(q(\hat{\theta}), \hat{\theta}) \frac{c_{q\theta}(q(\theta), \theta)}{c_{q\theta}(q(\hat{\theta}), \hat{\theta})} = 0 \tag{C3'}$$

at the optimum which is condition (C3). The argument is that otherwise there exists an implementable perturbation $h$ such that $G'(0) > 0$ which contradicts optimality. First, note that by the continuity of $q$ the left hand side of (C3') is continuous in $\theta$. Hence, if the left hand side of (C3') was not zero for some type, say for for the sake of argument it is positive, then there is an interval $(\theta'_1, \theta'_2)$ such that the left hand side of (C3') is positive for all $\theta \in (\theta'_1, \theta'_2)$ and their corresponding $\hat{\theta}$.

Let $h$ be zero for all $\theta \notin (\theta'_1, \theta'_2)$. Furthermore, let $h$ be continuous, strictly positive on $(\theta'_1, \theta'_2)$ and differentiable with bounded derivative on $(\theta'_1, \theta'_2)$.[39] Clearly, $G'(0) > 0$ in this case.

If the left hand side of (C3') is negative for some type, then again there has to be an interval $(\theta'_1, \theta'_2)$ such that the left hand side of (C3') is negative for all $\theta \in (\theta'_1, \theta'_2)$. In this case, choose $h$ to be zero for all $\theta \notin (\theta'_1, \theta'_2)$. Furthermore, let $h$ be continuous, strictly negative on $(\theta'_1, \theta'_2)$ and differentiable with bounded derivative on $(\theta'_1, \theta'_2)$. Again, $G'(0)$ is clearly positive.

It remains to show that such an $h$ is implementable. As $q^*$ is strictly increasing on $(\theta'_1, \theta'_2)$ and $h$ has a bounded derivative, $q$ is strictly increasing on $(\theta'_1, \theta'_2)$ for $\varepsilon$ small enough.

As $q^*$ is implementable and strictly increasing, $\hat{q}(\theta, q^*(\theta))$ is strictly decreasing on $(\theta'_1, \theta'_2)$ according to lemma 5. The boundedness of $h_\theta$ implies that $\hat{q}$ is also strictly decreasing $(\theta'_1, \theta'_2)$ for $\varepsilon > 0$ small enough: For types where $q^*$ is differentiable $\hat{q}_\theta(\theta, q(\theta)) = [\int_{\hat{q}(\theta, q(\theta))}^{q(\theta)} c_{q\theta\theta}(y, \theta) \, dy + q_\theta(\theta) c_{q\theta}(q(\theta), \theta)] / [c_{q\theta}(\hat{q}(\theta, q(\theta)), \theta)]$ which converges to the derivative of $\hat{q}(\theta, q^*(\theta))$ as $\varepsilon \to 0$. Hence, $d\,\hat{q}(\theta, q(\theta))/d\theta < 0$ at types where $q^*$ is differentiable for $\varepsilon > 0$ small enough. As $q^*$ is monotone, it is differentiable almost everywhere. Hence, strict monotonicity of $\hat{q}$ (for $\varepsilon > 0$ small enough) follows from continuity of $\hat{q}$ and the limit result for $d\,\hat{q}/d\theta$. The strict monotonicity of $\hat{q}$ together with $c_{qq\theta} < 0$ implies that $\phi(\theta, q(\theta))$ is strictly decreasing in $\theta$ (for $\varepsilon > 0$ small enough). Hence, $q$ is strictly increasing on $(\hat{\theta}_1, \hat{\theta}_2)$ for $\varepsilon > 0$ small enough.[40] That is, the monotonicity constraint is satisfied by $q$ for $\varepsilon > 0$ small enough.

It remains to check non-local incentive constraints under $q$ for $\varepsilon > 0$ small enough. Lemma 5 implies that $\Phi(\theta, \hat{\theta}) \geq 0$ for $\theta > \theta_2$ and $\hat{\theta} \in (\hat{\theta}_1, \theta_1)$: For these types, (NLIC) was satisfied with strict inequality under $q^*$ and therefore the same will be true for $q$ if $\varepsilon > 0$ is small enough because $\Phi(\theta, \hat{\theta})$ is continuous in $\varepsilon$.[41] The same argument holds for types $\theta \in (\hat{\theta}_1, \theta_1)$ and $\hat{\theta} < \hat{\theta}_2$. For $\theta$ and $\hat{\theta}$ such that $\hat{\theta}_1 < \hat{\theta} < \theta < \theta_1$, the perturbation does not affect $\Phi(\theta, \hat{\theta})$. Hence, (NLIC) is satisfied for these types.

Now take $\theta > \theta_2$ and $\hat{\theta} < \hat{\theta}_2$. Here it is important to note that $h(\theta_1) = h(\theta_2) = 0$. This implies that $\Phi(\theta_2, \hat{\theta}_2) = 0$ both under the original and the perturbed decision. As $\Phi(\theta, \hat{\theta}) = \Phi(\theta_2, \hat{\theta}_2) + \int_{\hat{\theta}}^{\theta} \int_{q(\hat{\theta})}^{min(q(\hat{\theta}_2), q(x))} -c_{q\theta}(y, x) \, dy \, dx + \int_{\theta_2}^{\theta} \int_{q(\hat{\theta}_2)}^{q(x)} -c_{q\theta}(y, x) \, dy \, dx = \Phi^*(\theta_2, \hat{\theta}_2) + \int_{\hat{\theta}}^{\theta} \int_{q(\hat{\theta})}^{min(q(\hat{\theta}_2), q(x))} -c_{q\theta}(y, x) \, dy \, dx + \int_{\theta_2}^{\theta} \int_{q(\hat{\theta}_2)}^{q(x)} -c_{q\theta}(y, x) \, dy \, dx = \Phi^*(\theta, \hat{\theta})$, (NLIC) holds between $\theta$ and $\hat{\theta}$.

Next take $\theta \in [\theta_1, \theta_2]$. I will distinguish four cases. First, take $\hat{\theta}_1 < \hat{\theta} < \theta_1$. Then $\Phi(\theta, \hat{\theta}) = \Phi(\theta_1, \hat{\theta}) + \int_{\theta_1}^{\theta} \int_{q(\hat{\theta})}^{q(x)} -c_{q\theta}(y, x) \, dy \, dx \geq \Phi(\theta_1, \hat{\theta}) = \Phi^*(\theta_1, \hat{\theta}) \geq 0$ where the first inequality holds because

[39] For example, $h(\theta) = -\left(\theta - \frac{\theta'_1 + \theta'_2}{2}\right)^2 + \left(\frac{\theta'_1 - \theta'_2}{2}\right)^2$ for $\theta \in (\theta'_1, \theta'_2)$ and $h(\theta) = 0$ for all other types.

[40] To reduce the length of the expressions, I use the notation $\hat{\theta}_i = \phi(\theta_i, q(\theta_i))$.

[41] To be precise, $\Phi^*(\theta, \hat{\theta})$ (where $\Phi^*$ is $\Phi$ under $q^*$) has to be bounded away from 0 for these types: Otherwise, that is if there was a sequence of $(\theta_n, \hat{\theta}_n)$ such that $\Phi^*(\theta_n, \hat{\theta}_n) \to 0$, for the limit types of (a subsequence of) this sequence (NLIC) would hold with equality as $\Phi^*$ is continuous. This would violate lemma 5.

$\int_{q(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy \geq \int_{q(\phi(x,q(x))}^{q(x)} -c_{q\theta}(y,\theta_1)\,dy = 0$ for all $x \in [\theta_1,\theta]$ by the monotonicity of $q$. Second, take $\hat{\theta} \in [\hat{\theta}_2,\hat{\theta}_1]$ and let $\theta' \in [\theta_1,\theta_2]$ be the type such that $\phi(\theta',q(\theta')) = \hat{\theta}$. Then $\Phi(\theta,\hat{\theta}) = \Phi(\theta',\hat{\theta}) + \int_{\theta'}^{\theta}\int_{q(\hat{\theta})}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx \geq \Phi(\theta',\hat{\theta}) = 0$ where the inequality holds because $\hat{q}$ is decreasing in $\theta$ and $\int_{q(\phi(x,q(x))}^{q(x)} c_{q\theta}(y,\theta_1)\,dy = 0$ for all $x \in [\theta_1,\theta_2]$. Third, take $\hat{\theta} < \hat{\theta}_2$. Then $\Phi(\theta,\hat{\theta}) = \Phi(\theta_2,\hat{\theta}) + \int_{\theta}^{\theta_2}\int_{q(\hat{\theta})}^{q(x)} c_{q\theta}(y,x)\,dy\,dx + \geq \Phi(\theta_2,\hat{\theta}) = \Phi(\theta_2,\hat{\theta}_2) + \int_{\hat{\theta}}^{\theta_2}\int_{q(\hat{\theta})}^{min(q(x),q(\hat{\theta}_2))} -c_{q\theta}(y,x)\,dy\,dx = \Phi^*(\theta_2,\hat{\theta}) \geq 0$ where I utilize $\Phi(\theta_2,\hat{\theta}_2) = \Phi^*(\theta_2,\hat{\theta}_2) = 0$ (recall that $h(\theta_2) = 0$). Fourth, $\hat{\theta} \in [\theta_1,\theta)$. One can choose $\theta_1$ and $\theta_2$ close enough (or equivalently let $h$ be non-zero only on a small enough subinterval of $(\theta_1,\theta_2)$) such that $c_{q\theta}(q(\theta_1),\theta_2) < 0$ which implies that (NLIC) is slack for $\theta,\hat{\theta} \in [\theta_1,\theta_2]$.

Last, take $\hat{\theta} \in [\hat{\theta}_2,\hat{\theta}_1]$ and let $\theta'$ be the type such that $\phi(\theta',q(\theta')) = \hat{\theta}$. First, $\theta \in (\hat{\theta}_1,\theta_1)$. Note that $\int_x^{\theta} c_{q\theta}(q(x),z)\,dz < 0$ for all $x \in [\hat{\theta},\hat{\theta}_1]$ because $\theta < \theta''(x)$ where $\theta''(x)$ is the type in $[\theta_1,\theta_2]$ such that $x = \phi(\theta'',q(\theta''))$. This implies that $\Phi(\theta,\hat{\theta}) \geq \Phi(\theta,\hat{\theta}_1) = \Phi^*(\theta,\hat{\theta}_1) \geq 0$. Second, $\theta \in [\theta_1,\theta_2]$ was already dealt with above. Third, $\theta > \theta_2$. Note that $\int_x^{\theta} c_{q\theta}(q(x),z)\,dz > 0$ for all $x \in (\hat{\theta}_2,\hat{\theta})$ as $\theta > \theta_2$. This implies $\Phi(\theta,\hat{\theta}) \geq \Phi(\theta,\hat{\theta}_2) = \Phi(\theta_2,\hat{\theta}_2) + \int_{\theta_2}^{\theta}\int_{q(\hat{\theta}_2)}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx = \Phi^*(\theta_2,\hat{\theta}_2) + \int_{\theta_2}^{\theta}\int_{q(\hat{\theta}_2)}^{q(x)} -c_{q\theta}(y,x)\,dy\,dx = \Phi^*(\theta,\hat{\theta}_2) \geq 0$. Fourth, $\theta \in [\hat{\theta}_2,\hat{\theta})$. One can choose $\theta_1$ and $\theta_2$ close enough (or equivalently let $h$ be non-zero only on a small enough subinterval of $(\theta_1,\theta_2)$) such that $c_{q\theta}(q(\hat{\theta}_2),\hat{\theta}_1) < 0$ which implies that (NLIC) is slack for $\theta,\hat{\theta} \in [\hat{\theta}_2,\hat{\theta}_1]$.

*Proof of (C3) similar to proof of theorem 1*

The proof works the following way: If (C3) did not hold for a pair of types $\theta',\hat{\theta}' \in (\underline{\theta},\bar{\theta})$ such that $\Phi^*(\theta',\hat{\theta}') = 0$, then a variation of the solution could be constructed which would (i) increase the principal's payoff and (ii) is incentive compatible. Since this contradicts the optimality of the solution, (C3) has to hold. The proof assumes that the solution is continuous and strictly monotone. Since (C3) is only used in section 5, this is all we need. Note, however, that the proof only utilizes that $q^*$ is continuous and strictly monotone in open neighborhoods of $\hat{\theta}'$ and $\theta'$ and is therefore more general.

I use the same notation as in the previous subsection: The solution is denoted by $q^*$ and a variation is denoted by $q = q * +\varepsilon h$. A variation is implementable if $q$ is incentive compatible for $\varepsilon > 0$ small enough. By $\Phi^*$ I denote $\Phi$ under the solution $q^*$.

To start with, note that for every type from/to which (NLIC) binds, there is another type arbitrarily close from/to which (NLIC) binds: Suppose otherwise, e.g. (NLIC) binds to $\hat{\theta}$ but there is an $\varepsilon' > 0$ such that (NLIC) is slack for all types in $(\hat{\theta}-\varepsilon',\hat{\theta})$ and also for all types in $(\hat{\theta},\hat{\theta}+\varepsilon')$. By theorem 1, $\eta$ is then constant on $(\hat{\theta}-\varepsilon',\hat{\theta})$ as well as on $(\hat{\theta},\hat{\theta}+\varepsilon')$. The continuity of $q^*$ (and therefore $\eta$) establishes then that $\eta$ is constant on $(\hat{\theta}-\varepsilon',\hat{\theta}+\varepsilon')$ and therefore (NLIC) does not bind from $\hat{\theta}$.

It follows that it is enough to establish (C3) for types $\theta'$, $\hat{\theta}'$ with $\Phi^*(\theta',\hat{\theta}') = 0$ such that (NLIC) binds for types arbitrarily close to $\theta'$, $\hat{\theta}'$. As soon as this is established, theorem 1 (almost directly) implies that (C3) also holds for all other types $\theta''$, $\hat{\theta}''$ such that $\Phi^*(\theta'',\hat{\theta}'') = 0$: Since (C3) holds for all type pairs from/to which (NLIC) binds and $\eta$ is constant on intervals of types where (NLIC) is slack, $\eta(\theta'') = \eta(\hat{\theta}'')$.

Now take interior types $\theta'$, $\hat{\theta}'$ such that (NLIC) binds from $\theta'$ to $\hat{\theta}'$ and such that (NLIC) binds to types arbitrarily close to $\hat{\theta}'$. Suppose (C3) does not hold. For concreteness, suppose $\eta(\theta') > \eta(\hat{\theta}')$ (the proof for the opposite case works similarly).[42] By the continuity of $q$, $\eta$ is continuous and therefore there exists an $\tilde{\eta}$ such that $\eta(\theta) > \tilde{\eta} > \eta(\hat{\theta})$ for all $\hat{\theta}$, $\theta$ in some open neighborhood of $\hat{\theta}'$, $\theta'$.

Let $\theta_1$ and $\theta_2$ be such that (i) $\theta_1 < \theta' < \theta_2$, (ii) (NLIC) binds from $\theta_1$ to $\hat{\theta}_1$, (iii) (NLIC) binds from $\theta_2$ to $\hat{\theta}_2$, (iv) $\eta(\theta) > \tilde{\eta}$ for all $\theta \in [\theta_1,\theta_2]$ and $\eta(\hat{\theta}) < \tilde{\eta}$ for all $\hat{\theta} \in [\hat{\theta}_2,\hat{\theta}_1]$ and (v) $\Phi^*(\theta_1,\hat{\theta}_2) > 0$. From the

[42]More precisely, the proof has to be adjusted at two places if $\eta(\theta') < \eta(\hat{\theta}')$: First, take $-h$ instead of $h$. Second, let $q(\hat{\theta}) = q(\tilde{\theta})$ for $\hat{\theta} \in (\hat{\theta}_2,\tilde{\theta})$.

discussion above it follows that such $\hat{\theta}_2$, $\hat{\theta}_1$, $\theta_1$, $\theta_2$ can be found.[43]

Now take $\theta_2'$ such that (i) $\theta_1 < \theta_2' < \theta_2$ and (ii) $\Phi^*(\theta, \hat{\theta}_2) \geq \delta$ for all $\theta \in [\theta_1, \theta_2']$ and some $\delta > 0$. By the continuity of $q^*$ and $\Phi(\theta_1, \hat{\theta}_2) > 0$ such a type can be found.

Consider the following variation $q$ of the optimal solution $q^*$: If $\theta \in [\theta_1, \theta_2']$, then $q(\theta) = q^*(\theta) + \varepsilon h(\theta)$ where[44]

$$h(\theta) = -\left(\theta - \frac{\theta_1 + \theta_2'}{2}\right)^2 + \left(\frac{\theta_1 - \theta_2'}{2}\right)^2$$

which is strictly positive on $(\theta_1, \theta_2')$ while $h(\theta_1) = h(\theta_2') = 0$. For a given $\varepsilon > 0$, let the variation for types in $[\hat{\theta}_2, \hat{\theta}_1]$ be such that

$$q(\hat{\theta}) = \begin{cases} q^*(\hat{\theta}) & \text{if } \hat{\theta} > \tilde{\theta} \\ q(\hat{\theta}_2) & \text{if } \hat{\theta} \leq \tilde{\theta} \end{cases}$$

where $\tilde{\theta}$ is chosen such that

$$\int_{\hat{\theta}_2}^{\hat{\theta}_1} \int_{q(x)}^{q^*(x)} c_{q\theta}(y, x)\, dy\, dx = \int_{\theta_1}^{\theta_2'} \int_{q^*(x)}^{q(x)} c_{q\theta}(y, x)\, dy\, dx. \tag{WA.8}$$

Clearly, a unique $\tilde{\theta} \in [\hat{\theta}_2, \hat{\theta}_1]$ exists for a given $\varepsilon > 0$ small enough and $\tilde{\theta}$ continuously approaches $\hat{\theta}_2$ as $\varepsilon \to 0$ (as $q^*$ is strictly increasing and continuous). The variation is depicted in figure WA.3. Rearranging (WA.8) gives

$$\int_{\hat{\theta}_2}^{\hat{\theta}_1} \int_{q(x)}^{q^*(x)} c_{q\theta}(y, x)\, dy\, dx - \int_{\theta_1}^{\theta_2'} \int_{q^*(x)}^{q(x)} c_{q\theta}(y, x)\, dy\, dx = 0 \tag{WA.9}$$

for any given $\varepsilon \geq 0$ and therefore the derivative of the left hand side of the last equation with respect to $\varepsilon$ is zero.

For all other types we take $q = q^*$. Clearly, $q$ is monotonically increasing for $\varepsilon > 0$ small enough because $q^*$ is strictly increasing (and $h$ has a bounded derivative).

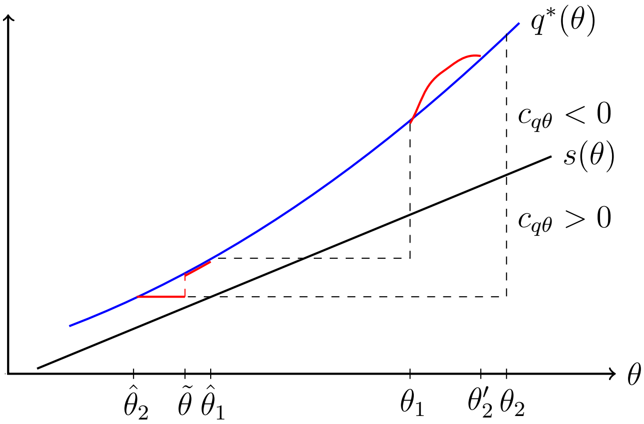


Figure WA.3: Variation for proof (C3)

[43]Note that the fifth requirement follows directly from $\hat{\theta}_2 \neq \hat{\theta}_1$ and the assumption that $q$ is continuous and strictly increasing: Because of $\Phi^*(\theta_1, \hat{\theta}_1) = 0$, (C1) must hold. But then $\Phi^*(\theta_1 + \varepsilon', \hat{\theta}_2) < 0$ for some $\varepsilon' > 0$ if we had $\Phi^*(\theta_1, \hat{\theta}_2) = 0$.

[44]This is only one specific example. Any differentiable $h$ with bounded derivative and $h(\theta_1) = h(\theta_2') = 0$ while $h(\theta) > 0$ on $(\theta_1, \theta_2')$ will work as well.

The change in the principal's payoff due to the variation is

$$\begin{aligned}\Delta(\varepsilon) &= \int_{\hat{\theta}_2}^{\hat{\theta}_1}\int_{q(x)}^{q^*(x)}[u_q(y,x)-c_q(y,x)]f(x)+(1-F(x))c_{q\theta}(y,x)\,dy\,dx\\ &\quad-\int_{\theta_1}^{\theta_2'}\int_{q^*(x)}^{q(x)}[u_q(y,x)-c_q(y,x)]f(x)+(1-F(x))c_{q\theta}(y,x)\,dy\,dx.\end{aligned}$$

Now consider the derivative of $\Delta$ at $\varepsilon=0$. Using theorem 1, we can write this as

$$\begin{aligned}\Delta'(0) &= -\int_{\hat{\theta}_2}^{\hat{\theta}_1}\eta(x)c_{q\theta}(q^*(x),x)\left.\frac{\partial q(x)}{\partial\varepsilon}\right|_{\varepsilon=0}dx-\int_{\theta_1}^{\theta_2'}\eta(x)c_{q\theta}(q^*(x),x)\left.\frac{\partial q(x)}{\partial\varepsilon}\right|_{\varepsilon=0}dx\\ &> \tilde{\eta}\left[\int_{\hat{\theta}_2}^{\hat{\theta}_1}c_{q\theta}(q^*(x),x)\left.\frac{\partial q(x)}{\partial\varepsilon}\right|_{\varepsilon=0}dx-\int_{\theta_1}^{\theta_2'}c_{q\theta}(q^*(x),x)\left.\frac{\partial q(x)}{\partial\varepsilon}\right|_{\varepsilon=0}dx\right]\\ &= 0\end{aligned}$$

where the inequality follows from $\eta(\theta)>\tilde{\eta}$ for all $\theta\in[\theta_1,\theta_2)$ and $\eta(\hat{\theta})<\tilde{\eta}$ for all $\hat{\theta}\in[\hat{\theta}_2,\hat{\theta}_1]$ (and the obvious fact that $q$ is increasing in $\varepsilon$ for $\theta\in(\theta_1,\theta_2')$ while it is decreasing in $\varepsilon$ for $\hat{\theta}\in(\hat{\theta}_2,\hat{\theta}_1)$). The last term is then zero because it is the derivative of the left hand side of (WA.9) with respect to $\varepsilon$ (at $\varepsilon=0$) which is zero by construction of the variation. Hence, the principal's payoff is higher in the variation for $\varepsilon>0$ small enough. This contradicts the optimality of $q^*$ provided that the variation is incentive compatible for $\varepsilon>0$ small enough. This will be shown next.

First note that, since $q^*>s$, $\hat{\theta}_2$, $\hat{\theta}_1$ can be chosen sufficiently close together such that $q^*(\hat{\theta}_2)\geq s(\hat{\theta}_2)$. This implies that $\Phi(\theta,\hat{\theta})>0$ for $\hat{\theta}_2\leq\hat{\theta}<\theta\leq\hat{\theta}_1$.

Second, we consider $\theta\in[\hat{\theta}_1,\theta_1]$ and $\hat{\theta}\in[\hat{\theta}_2,\hat{\theta}_1]$. Note that $\min_{\theta\in[\hat{\theta}_1,\theta_1]}\Phi^*(\theta,\hat{\theta}_2)>0$: Since $\Phi^*$ is continuous in its first argument the minimum exists. For $\theta\in(\hat{\theta}_1,\theta_1)$, $\Phi^*(\theta,\hat{\theta}_2)>0$ follows from the no-overlap property (lemma 5) (recall that $\Phi^*(\theta_1,\hat{\theta}_1)=0$). For the boundary types $\hat{\theta}_1$ and $\theta_1$ we have $\Phi^*(\theta,\hat{\theta}_2)>0$ because of the first step (previous paragraph) and the assumption $\Phi^*(\theta_1,\hat{\theta}_2)>0$ respectively. Consequently, $\min_{\theta\in[\hat{\theta}_1,\theta_1]}\Phi^*(\theta,\hat{\theta}_2)>0$. For $\theta\in[\hat{\theta}_1,\theta_1]$, this implies that $\Phi(\theta,\hat{\theta}_2)=\Phi^*(\theta,\hat{\theta}_2)-\Phi^*(\tilde{\theta},\hat{\theta}_2)\geq 0$ for $\varepsilon>0$ small enough because $\Phi^*(\theta,\hat{\theta}_2)$ is bounded away from zero and $\Phi^*(\tilde{\theta},\hat{\theta}_2)$ continuously approaches zero as $\varepsilon\to 0$. Since $\Phi(\theta,\hat{\theta})=\Phi(\theta,\hat{\theta}_2)$ for $\hat{\theta}\leq\tilde{\theta}$, we get that no type in $[\hat{\theta}_1,\theta_1]$ wants to misrepresent as a type below $\tilde{\theta}$ for $\varepsilon>0$ small enough. Note that $\Phi(\theta,\hat{\theta})=\Phi^*(\theta,\hat{\theta})$ for $\hat{\theta}>\tilde{\theta}$ which means that no type in $[\hat{\theta}_1,\theta_1]$ wants to misrepresent as a type above $\tilde{\theta}$ either.

Third, take $\theta\in[\theta_1,\theta_2']$ and $\hat{\theta}\in[\hat{\theta}_2,\hat{\theta}_1]$. By construction $\min_{\theta\in[\theta_1,\theta_2']}\Phi^*(\theta,\hat{\theta}_2)$ is bounded from below by $\delta>0$. Given that $\Phi(\theta,\hat{\theta}_2)\geq\Phi^*(\theta,\hat{\theta}_2)-\Phi^*(\tilde{\theta},\hat{\theta}_2)$ for $\theta\in[\theta_1,\theta_2']$, the same continuity argument as in the previous paragraph yields that no type in $[\theta_1,\theta_2']$ wants to misrepresent as $\hat{\theta}_2$ (or any other $\hat{\theta}\leq\tilde{\theta}$). Furthermore, $\Phi(\theta,\hat{\theta})\geq\Phi^*(\theta,\hat{\theta})$ for $\hat{\theta}>\tilde{\theta}$ which then implies that $\Phi(\theta,\hat{\theta})\geq 0$ for $\theta\in[\theta_1,\theta_2']$ and $\hat{\theta}\in[\hat{\theta}_2,\hat{\theta}_1]$ (for $\varepsilon>0$ sufficiently small).

All other incentive constraints are either relaxed or not affected by the variation. Consequently, the variation is incentive compatible which concludes the proof.