Game Theory
Perfect equilibrium in extensive form games

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Example

Information sets in a game tree are indicated by (dashed) lines between the nodes that build the information set.
Example

Here a game tree with a chance move.
Intermezzo: Imperfect recall I

Figure: extensive form games with imperfect recall
Intermezzo: Imperfect recall II

Interpretation 1:

- left game: P1 forgets whether he moved already or not
- right game: P1 forgets what he previously knew (which move chance played)
- bottom game: P1 forgets what he previously did

Interpretation 2:

- information set represents the information about history that is inherent in the game, i.e. the information the game provides to the player when it is his move
- the player can refine this information by inferences
- under this interpretation the expression “imperfect recall” is a bit misleading
how many pure strategies does P1 have in each of the three games on the previous slide?:

- left game:
- right game:
- bottom game:

imperfect recall is a bit troublesome (at least in the first interpretation)

we talk now a bit about it and then “forget it”

to illustrate the problems consider the sleeping beauty example
Example (sleeping beauty)

- on Sunday the sleeping beauty is put to sleep
- a fair coin is then tossed
  - if heads:
    - beauty is wakened and interviewed on Monday, and then the experiment ends
  - if tails:
    - beauty is wakened and interviewed on Monday and Tuesday
    - when she is put to sleep again on Monday, she is given a dose of an amnesia-inducing drug that ensures she cannot remember her previous awakening
    - experiment ends after she is interviewed on Tuesday

Any time Sleeping Beauty is wakened and interviewed, she is asked, “What probability do you assign to the event that the coin landed heads?”

What should she answer?
Mixed and behavioral strategies I

Definition

Let $\mathcal{I}_i$ be the set of all information sets in which player $i$ has to act and denote the actions from which player $i$ has to choose in information set $I_i$ by $A(I_i)$. A **behavioral strategy** of player $i$ is a collection $(\beta_i(I_i))_{I_i \in \mathcal{I}_i}$ of independent probability measures where $\beta_i(I_i)$ is a probability measure over $A(I_i)$.

- behavioral strategy prescribes a mix over actions at every information set a player might have to act.
First example game:

- Can you give an example of a behavioral strategy of P1?
Perfect equilibrium I

Figure: Selten’s example

How should rational play look in this example?
Perfect equilibrium II

- no proper subgames $\Rightarrow$ every NE is SPNE
- $(R_1, L_2, l)$ is NE but is it reasonable?
- would player 2 play $L_2$ if he was called upon to act?
- will player 1 then play $R_1$?
- $(R_1, L_2, l)$ is NE but not self enforcing!
- a self enforcing equilibrium:
  - P1 plays $L_1$ and $R_1$ with prob 1/2
  - P2 plays $R_2$
  - P3 plays 1/4 $l$ and 3/4 $r$
Perfect equilibrium III

- Problem with \((R_1, L_2, l)\): information set that is reached with zero probability when equilibrium strategies are played
- in corresponding strategic form: P2 plays weakly dominated strategy
- recall Selten’s idea of trembling hand perfect equilibrium: all info sets are reached with positive probability when trembling
- reminder: trembling hand perfection (roughly): a perfect equilibrium is a NE that is a limit of completely mixed equilibria
Perfect equilibrium IV

Figure: Why agent strategic form?

What is the SPNE? Is there another trembling hand perfect NE in the corresponding strategic form?
Perfect equilibrium V

- one SPNE: ((L,l),a)
- ((R,r),b): P1 does not consider that he himself could make the mistake of playing L instead of R
- \( \Rightarrow \) agent strategic form: for a given extensive form game G the agent strategic form
  - has a player \( i_h \) for each information set \( h \) of each player \( i \) in G
  - the actions of \( i_h \) correspond to the actions of \( i \) at \( h \) in G
  - the payoff function of \( i_h \) is equal to the payoff function of \( i \) in G
  - is a strategic form game!

- idea is fully decentralized decision making where each player delegates decision making at any information set to one agent
Perfect equilibrium VI

Figure: getting to the agent strategic form
Perfect equilibrium VII

Table: agent strategic form

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<thead>
<tr>
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<th>a</th>
<th>b</th>
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<tr>
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<td>0,2,0</td>
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<tr>
<td>R</td>
<td>1,1,1</td>
<td>1,1,1</td>
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</tr>
</tbody>
</table>

- strategies in the agent strategic form are similar to behavioral strategies in the extensive form game
Perfect equilibrium VIII

**Definition**

A (trembling hand) **perfect equilibrium of an extensive form game** $G$ is a perfect equilibrium in the agent strategic form of $G$.

- we already showed that every finite strategic form game has a perfect equilibrium this gives directly

**Corollary**

*There exists a perfect equilibrium in every finite extensive form game.*
Perfect equilibrium IX

- let’s go back to Selten’s example:
  - P2’s best action in each perturbed game is $R_2$ and therefore no equilibrium including $L_2$ is perfect
  - is the self-enforcing equilibrium in Selten’s example perfect? Yes: let the equilibrium strategies in the fully mixed perturbed game be
    - P1 plays $L_1$ with prob $1/(2 - \varepsilon)$ and $R_1$ with prob $(1 - \varepsilon)/(2 - \varepsilon)$
    - P2 plays $L_2$ with prob $\varepsilon$ and $R_2$ with prob $1 - \varepsilon$
    - P3 plays $l$ with prob $1/(4 \times (1 - \varepsilon))$ and $r$ with prob $(3 - 4\varepsilon)/(4 \times (1 - \varepsilon))$
Review questions

- What is imperfect recall and what problems does it cause?
- What is a behavioral strategy?
- Why do we define perfect equilibrium in extensive form games through the “agent strategic form” instead of using the usual normal form representation of the extensive form game?
- How is perfect equilibrium defined in extensive form games?

Reading: perfect equilibrium: MSZ 7.3.2 or OR 12.5.2; on extensive form games: OR 11.1 and 11.4
1. Bob has to drive home from work. To get home he has to go straight at the first crossroad and turn right at the second crossroad. The two crossroads look exactly identical and Bob is “absent minded”: Whenever he arrives at a crossroad he does not know whether it is the first or the second crossroad. Assume that his payoff from coming home is 1 while his payoff from going wrong is 0. What should Bob do?

2. Write down the corresponding normal form game of the game below. Show that there are four Nash equilibria and show that (CD,L) is a perfect equilibrium in the normal form game. Write down the agent strategic form and show that (CD,L) is not a perfect equilibrium in the agent strategic form. Find a perfect equilibrium of the game.