Welfare optimal information structures in bilateral trade

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Preliminary version!

Abstract

This paper analyzes the welfare maximizing information structure (and mechanism) in a bilateral trade setting. The welfare loss in the optimal information structure constitutes the minimal welfare loss due to asymmetric information. With binary underlying types it is shown that more than 95% of first best welfare can be achieved while the optimal mechanism without information design may achieve less than 90% of first best welfare. For more general type distributions, the optimal information structure is a monotone partition of the type space (but is not explicitly solved for) and the optimal mechanism is deterministic.

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1. Introduction

Asymmetric information can lead to inefficiencies in economic transactions. Myerson and Satterthwaite (1983) established this result in a bilateral trade setting in which the buyer (seller) has private information about his valuation (costs). They also derive the welfare maximizing mechanism for their setting. This paper extends their analysis by considering not only the welfare maximizing mechanism but also the welfare maximizing information structure. More precisely, imagine that buyer and seller do not know their own valuation and costs perfectly but only have a private noisy signal, i.e. an estimate, of these variables. This paper derives the information structure, i.e. a mapping from true valuation and costs to signals, that maximizes expected welfare. Welfare under this optimal information structure is consequently the maximal welfare that is attainable (by any information structure and mechanism) in light of asymmetric information. Hence, the difference with first best welfare constitutes the loss of welfare that can be attributed to asymmetric information while any additional welfare loss has to be blamed on bad institutions, i.e. either a bad mechanism or a bad information structure.

The goal of the paper is therefore to derive the optimal information structure in order to establish the welfare loss due to asymmetric information in bilateral trade.

Of course, there are also literal interpretations of information design. For example, conventions and institutions like the legal framework for contracting affect the information of the players. Many goods are indeed transacted at a time where value and costs are not entirely clear. Tickets for flights are, for instance, bought and sold long before the actual travel which implies that the true (fuel) costs of the flight are not perfectly known at the time of contracting. Similarly, tickets for music festivals are sold at a time at which the final line-up is still subject to changes which implies that neither costs

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nor valuation are perfectly known at the time of contracting. In other examples, buyers know only a set of key characteristics of the product but do not know all features when buying and more generally a seller's opportunity costs which depend on potential future buyers' valuations are uncertain. In this sense, the welfare optimal information structure gives an indication about the optimal point of time for contracting or the ideal set of known attributes.

Limiting the information of a player has several effects. Consider, for example, a buyer whose valuation is either high or low and suppose the information structure is such that he does not get any information about which of the two valuations has realized. This makes it impossible to establish whether his valuation is above or below the costs of the seller and therefore less information directly harms efficiency. On the other hand, giving a player less information also reduces his information rent. The latter effect relaxes the budget balance constraint and will therefore generally improve welfare. The following example illustrates how information design can improve on the optimal mechanism as derived in Myerson and Satterthwaite (1983).

1.1. A simple example

In the following example, the seller has costs of either $c_l = 1$ or $c_h = 6$ and both values are equally likely. The buyer has valuation $v_l = 2$ or $v_h = 8$ also with equal probabilities. If these valuations were private information of the respective player, first best – i.e. trade if and only if valuation is above cost – is infeasible. To see this, recall that in the welfare maximal mechanism the participation constraint of the worst type of each player holds with equality. Therefore the trading prices between type v_h and c_h is $t(v_h, c_h) = 6$ and $t(v_l, c_l) = 2$. To maintain incentive compatibility for type v_h, the following inequality has to hold: $8 - 6/2 - t(v_h, c_l)/2 \geq (8-2)/2 \Leftrightarrow t(v_h, c_l) \leq 4$. The incentive compatibility constraint for type c_l is $2/2 + t(v_h, c_l)/2 - 1 \geq (6-1)/2 \Leftrightarrow t(v_h, c_l) \geq 5$. As the two incentive compatibility constraints cannot be satisfied at the same time, the first best efficient mechanism is not feasible.

The trading probabilities of the second best mechanism are summarized in table 1. The transfers in the second best mechanism are $t(v_h, c_h) = 5$, $t(v_l, c_l) = 2$ and $t(v_h, c_l) = 26/5$.[1]

type	c_l	c_h
v_l	4/5	0
v_h	1	1

Table 1: Optimal trading probabilities when true types are private information

Now consider the following information structure: The buyer learns his valuation perfectly but the seller receives a binary signal which is either h (high) or l (low) according to the following distribution: if costs are c_h, the seller always receives signal h; if costs are low the signal is h with probability β and l with probability $1-\beta$. The expected costs when receiving the low signal are therefore 1 while they are $\tilde{c}_h = \frac{6+\beta}{1+\beta}$ when receiving the high signal. With these expected costs we want to trade if and only if the valuation is above expected costs using the transfers $t(v_h, \tilde{c}_h) = \frac{6+\beta}{1+\beta}$ and $t(v_l, c_l) = 2$. Note that the participation constraints hold in this case. Incentive compatibility holds for a buyer of type v_h if

$$8 - \frac{1+\beta}{2}\frac{6+\beta}{1+\beta} - \frac{1-\beta}{2}t(v_h, c_l) \geq \frac{1}{2}(1-\beta)(8-2)$$

[1] While a detailed derivation of the optimal mechanism is given at a later point one can almost guess the optimal mechanism in this example: The transfers $t(v_h, c_h) = 5$ and $t(v_l, c_l) = 2$ keep the participation constraints of the worst types with equality. As the trade between v_l and c_l yields the lowest welfare, it is clear that this is the one where the trading probability will be less than 1. The incentive compatibility constraint of v_h reads $8 - 6/2 - t(v_h, c_l)/2 \geq y(v_l, c_l)(8-2)/2$ and the one of c_l is $(t(v_h, c_l) - 1)/2 + y(v_l, c_l)(2-1)/2 \geq (6-1)/2$ where $y(v_l, c_l)$ is the probability of trade between type v_l and c_l. The optimal values of $t(v_h, c_l)$ and $y(v_l, c_l)$ are such that both constraints hold with equality.

and for a seller of type c_l if

$$\frac{1}{2}(t(v_h, c_l) - 1) + \frac{1}{2}(2-1) \geq \frac{1}{2}\left(\frac{6+\beta}{1+\beta} - 1\right).$$

The lowest value of β compatible with these two conditions is $\beta = -7/5 + \sqrt{54}/5 \approx 0.06969$ together with $t(v_h, c_l) = (6+\beta)/(1+\beta) - 1 \approx 4.67423$ (both constraints hold with equality for these values) which leads to $\tilde{c}_h \approx 5.67423$.

The trading probabilities in terms of the true types and signals are given in table 2. A seller with cost c_l trades with a buyer of type v_l only if he receives signal l and therefore only with probability $1-\beta$. All other values should be self explanatory. Two things are noteworthy: First, information design strictly increases welfare by increasing the probability of trade between v_l and c_l. Second, we use the first best trading rule (trade if and only if expected value is above expected costs where expectations are taken conditional on the signal) with the information structure derived above.

type	c_l	c_h
v_l	≈ 0.93031	0
v_h	1	1

signal	l	h
v_l	1	0
v_h	1	1

Table 2: Optimal trading probabilities for true types (left) and signals (right) when the seller has only a noisy signal of his type

To understand the mechanics behind the example, compare the effect of a direct reduction of the trading probability with privately known types and the reduction in trading probability through the introduction of the noisy signal on the binding incentive compatibility constraints of v_h and c_l. First consider v_h. Reducing the probability of trade between v_l and c_l makes a misrepresentation as type v_l less profitable. This effect is true for both methods. The noisy signal, however, also leads to a reduction of $t(v_h, c_h)$ from c_h to $\tilde{c}_h$ which makes truthtelling more attractive for type v_h. Hence, a smaller reduction of the trading probability is needed with noisy signals than with perfectly known types to satisfy v_h's incentive compatibility constraint.

Second, consider the incentive compatibility constraint of c_l. Reducing the trading probability of v_l and c_l by β reduces his expected utility from truthtelling by $(v_l - c_l)\beta/2$ (no matter whether this reduction is due to noisy signals or not). With perfectly known types this negative effect has to be made up by the positive effect on v_h's incentive constraint described above. With noisy signals, there is an additional effect relaxing c_l's incentive constraint: As $t(v_h, c_h) = \tilde{c}_h < c_h$, the deviation to c_h is less attractive, i.e. a reduction of the trading probability between v_l and c_l by β using noisy signals reduces $t(v_h, c_h)$ by $5\beta/(1+\beta)$ and therefore the expected utility of deviating to h by $5\beta/(1+\beta)/2$. That is, increasing β relaxes overall the incentive compatibility constraint of l types with noisy signals while a higher β makes c_l's incentive compatibility constraint more demanding in the setting with perfectly known types.

It will be shown that the example here is typical. In particular for a binary distribution of costs and values the optimal information structure will always be fully informative for one player while the other player's information structure is binary and one signal reveals his "good type" perfectly. The optimal mechanism enforces trade with probability 1 unless both players receive the "bad signal" in which case trade has probability 0. Under the welfare optimal information structure and binary type distributions trade takes place if and only if the buyer signal exceeds the seller signal.

1.2. Literature

The setting is similar to Myerson and Satterthwaite (1983) who derive the welfare optimal mechanism in a bilateral trade setting in which (i) trade is voluntary, (ii) the budget has to be balanced and (iii) the buyer (seller) privately knows his valuation (costs). Keeping (i) and (ii) this paper changes (iii) by deriving the information structure that maximizes expected welfare. Note that valuations and costs are independently distributed in Myerson and Satterthwaite (1983). Following the arguments of Cremer and McLean (1988), correlated types would allow to extract this private information at no cost and therefore allow for full efficiency. (In fact, telling each player both valuation and cost is one correlated information structure that eliminates private information altogether, see appendix A.) Hence, information design could achieve first best welfare if the signal structures of buyer and seller were correlated. Hence, this paper extends the assumption in Myerson and Satterthwaite (1983) that types are independent by requiring that also signals of buyer and seller have to be independent.

This paper is related to a recent literature on information design as surveyed in Bergemann and Morris (2019). Within this literature the following papers consider bilateral trade settings. Roesler and Szentes (2017) determine the consumer surplus maximizing information structure if a monopolist seller makes a take-it-or-leave-it offer. The main differences to this paper are (i) the objective (consumer surplus vs. expected welfare), (ii) the seller has private information in this paper while he does not in Roesler and Szentes (2017) and (iii) the use of welfare optimal mechanism instead of a take-it-or-leave-it offer.[2] Technically, closest is Bergemann and Pesendorfer (2007) which derives the independent information structures that maximize revenue in independent private value auctions. As in this paper, the optimal information structure turns out to be a finite monotone partition. Apart from the objective (revenue vs. welfare) and the setting (auction vs. bilateral trade), the main difference is the presence of budget balance as an additional constraint in my setup. While some proofs are similar in style to proofs in Bergemann and Pesendorfer (2007), the presence of this constraint complicates matters significantly. A technical contribution of this paper to the literature on information design is indeed the addition of this budget balance constraint in a setting with two-sided asymmetric information. Finally, Lang (2016, ch. 4) shows by means of an example that welfare can be higher if players have coarse information in a bilateral trade setting. However, he does not analyze welfare maximizing information structures.

2. Model

A single indivisible object may be traded between a buyer and a seller. The buyer's valuation for the object is distributed according to the cumulative distribution function (cdf) H_B with support on a bounded subset of $\mathbb{R}_+$. The buyer maximizes a linear utility function, i.e. he maximizes expected valuation minus expected payments. The seller's (opportunity) costs for making the object are distributed according to cdf H_S (on a bounded subset of $\mathbb{R}_+$) and the seller maximizes expected payments minus expected costs. Welfare equals valuation minus costs if trade takes place and zero otherwise.

To make the problem interesting, I assume that the supports of H_B and H_S are overlapping and that for each type there are strictly positve gains from trade with some types of the other player:

Assumption 1 (Overlapping support)**.** $\min supp(H_S) < \min supp(H_B) < \max supp(H_S) < \max supp(H_B)$

A signal structure for the buyer maps from the support of H_B to a probability distribution over

[2]Condorelli and Szentes (2018) analyze a hold-up problem that can be viewed as a buyer choosing his distribution of valuations and privately learning his valuation after which a seller who only knows the distribution sets a profit maximizing price. In contrast to Roesler and Szentes (2017), the buyer is not restricted to choosing an information structure on a given "true type distribution" but can in fact choose the "true type distribution" itself. This difference to my paper is in addition to those already stated in the text above for Roesler and Szentes (2017).

a set of signals $\Sigma_v \subset \mathbb{R}$. As the buyer cares only about his expected valuation it is without loss of generality to identify a signal with the expected valuation it induces. Hence, a signal v is understood to imply that the buyer has expected valuation v when receiving this signal. With this convention, a signal structure can be described by a probability distribution over a set of expected valuations. A signal structure F is then feasible if and only if H_B is a mean preserving spread of F. The same applies to the seller: A feasible signal structure for the seller can be described by a distribution G over costs such that H_S is a mean preserving spread of G. A signal structure is then described by a feasible F and a feasible G. Note that the two distributions F and G are required to be independent as otherwise first best could be achieved easily by essentially eliminating the information asymmetry between buyer and seller, see appendix A.

Without loss of generality only incentive compatible direct revelation mechanisms are considered. A direct revelation mechanism ("mechanism" in the following) assigns to each pair of signals (v, c) a probability of trade $y(v, c) \in [0, 1]$ and a transfer $t_B(v, c) \in \mathbb{R}$ the buyer pays as well as a transfer $t_S(v, c) \in \mathbb{R}$ the seller receives. Incentive compatibility means that truthfully revealing his signal must yield a higher expected utility for a player than announcing another signal given that the other player announces his signal truthfully, i.e.

$$\int_{\mathbb{R}} vy(v,c) - t_B(v,c)\, dG(c) \geq \int_{\mathbb{R}} vy(v',c) - t_B(v',c)\, dG(c) \quad \text{for all } v' \in supp(F) \tag{ICB}$$

$$\int_{\mathbb{R}} t_S(v,c) - cy(v,c)\, dF(v) \geq \int_{\mathbb{R}} t_S(v,c') - cy(v,c')\, dF(v) \quad \text{for all } c' \in supp(G). \tag{ICS}$$

Participation is voluntary at the interim stage: in order to be feasible a mechanisms must not only be incentive compatible but also yield an expected utility of at least zero conditional on any signal. Denoting the buyer's (seller's) interim utility by U (Π), this can be written as

$$U(v) = \int_{\mathbb{R}} vy(v,c) - t_B(v,c)\, dG(c) \geq 0 \qquad \text{for all } v \in supp(F) \tag{PCB}$$

$$\Pi(c) = \int_{\mathbb{R}} t_S(v,c) - cy(v,c)\, dF(v) \geq 0 \qquad \text{for all } c \in supp(G). \tag{PCS}$$

Ex post budget balance requires $t_B(v, c) \geq t_S(v, c)$ but as will be explained later it is without loss of generality to use the weaker ex ante budget balance constraint instead

$$\int_{\mathbb{R}} \int_{\mathbb{R}} t_B(v,c) - t_S(v,c)\, dF(v)\, dG(c) \geq 0.$$

Note that due to the convention that signals are the corresponding expected values, expected welfare equals $y(v, c)(v - c)$ if trade takes place between a buyer with signal v and a seller with signal c. The objective of this paper is to find the feasible information structure (F and G) and feasible mechanism (y, t_B and t_S) that maximize expected welfare given true type distributions H_B and H_S.

In the following I will refer to an element of the support of H_B or H_S as *type* and to an element of the support of F or G as *signal*. I will call the buyer's (seller's) signal structure *fully informative* if $F = H_B$ ($G = H_S$) and *noisy* otherwise.

3. Welfare optimal mechanism with given types and discrete type distributions

This section presents the optimal mechanism for a given discrete information structure. Let the buyer have signal $v_i \in \{v_1, \ldots, v_n\}$ with probability ω_i and the seller have signal $c_j \in \{c_1, \ldots, c_m\}$ with probability γ_j. Lower indices are assumed to denote lower signals. Expected welfare equals

$$\sum_{i=1}^{n}\sum_{j=1}^{m} y(v_i, c_j)(v_i - c_j)\omega_i\gamma_j \tag{1}$$

where $y(v_i, c_j)$ is the probability of trade for v_i and c_j. A buyer of signal v_i has expected utility

$$U(v_i) = \sum_{j=1}^{m}(v_i y(v_i, c_j) - t_B(v_i, c_j))\gamma_j = v_i Y_B(v_i) - T_B(v_i) \tag{2}$$

where the expected transfer $\sum_j t_B(v_i, c_j)\gamma_j$ is denoted by $T_B(v_i)$ and the expected probability of trade is denoted by $Y_B(v_i) = \sum_j y(v_i, c_j)\gamma_j$. Similarly, the expected utility of the seller is

$$\Pi(c_j) = \sum_{i=1}^{n}(t_S(v_i, c_j) - c_j y(v_i, c_j))\omega_j = T_S(c_j) - c_j Y_S(c_j). \tag{3}$$

The goal is to determine the welfare maximizing y and transfer rules t_S and t_B subject to

- the individual rationality constraints

$$U(v_i) \geq 0 \quad \text{for all } v_i \in \{v_1, \ldots, v_n\} \qquad \Pi(c_j) \geq \ \text{for all } c_j \in \{c_1, \ldots, c_m\}, \tag{IR}$$

- the incentive compatibility constraints

$$v_i Y_B(v_i) - T_B(v_i) \geq v_i Y_B(v_k) - T_B(v_k) \quad \text{for all } v_i, v_k \in \{v_1, \ldots, v_n\}, \tag{IC$_\text{B}$}$$

$$T_S(c_j) - c_j Y_S(c_j) \geq T_S(c_k) - c_j Y_S(c_k) \quad \text{for all } c_j, c_k \in \{c_1, \ldots, c_m\}, \tag{IC$_\text{S}$}$$

- budget balance

$$\sum_{i=1}^{n}\omega_i T_B(v_i) = \sum_{j=1}^{m}\gamma_j T_S(c_j). \tag{BB}$$

It is straightforward to show that in this setting every ex ante budget balanced mechanism can be made ex post budget balanced in the sense that starting from an ex ante budget balanced mechanism one can manipulate the transfer rules (without changing the decision rule y) in a way that the new mechanism satisfies ex post budget balance (and IC as well as IR constraints are nor affected). The proof of this is standard and given in appendix B. For this reason, it is without loss of generality to use the simpler ex ante budget balance condition instead of its ex post version.

In particular, this means that the objective and all constraints can be expressed in terms of interim transfers T_B and T_S or alternatively in terms of interim rents U and Π. The following lemma gives a simple characterization of incentive compatibility for the discrete case.

Lemma 1. (IC$_\text{B}$) *is satisfied if and only if Y_B is increasing and*

$$U(v_i) = U(v_{i-1}) + \tilde{Y}_B(v_{i-1})(v_i - v_{i-1}) \quad \textit{for} \quad i = 2, \ldots, n \tag{4}$$

where $Y_B(v_{i-1}) \leq \tilde{Y}_B(v_{i-1}) \leq Y_B(v_i)$. (IC$_\text{S}$) *is satisfied if and only if Y_S is decreasing and*

$$\Pi(c_j) = \Pi(c_{j+1}) + \tilde{Y}_S(c_j)(c_{j+1} - c_j) \tag{5}$$

where $Y_S(c_j) \geq \tilde{Y}_S(c_j) \geq Y_S(c_{j+1})$.

(4) and (5) can be rewritten as[3]

$$
\begin{aligned}
U(v_i) &= U(v_1) + \sum_{k=1}^{i-1} \tilde{Y}_B(v_k)(v_{k+1} - v_k) \\
\Pi(c_j) &= \Pi(c_m) + \sum_{k=j}^{m-1} \tilde{Y}_S(c_k)(c_{k+1} - c_k).
\end{aligned}
$$

This allows to rewrite the budget balance constraint (BB) as

$$
-U(v_1)+\sum_{i=1}^{n} \omega_i \left[v_i Y_B(v_i) - \sum_{k=1}^{i-1} \tilde{Y}_B(v_k)(v_{k+1} - v_k) \right] \geq \Pi(c_m)+\sum_{j=1}^{m} \gamma_j \left[c_j Y_S(c_j) + \sum_{k=j}^{m-1} \tilde{Y}_S(c_k)(c_{k+1} - c_k) \right]
$$

which is equivalent to[4]

$$
\sum_{i=1}^{n} \left[\omega_i Y_B(v_i) v_i - (v_{i+1} - v_i) \tilde{Y}_B(v_i)(1 - W_i) \right] \geq U(v_1)+\Pi(c_m)+\sum_{j=1}^{m} \left[\gamma_j Y_S(c_j) c_j + (c_{j+1} - c_j) \tilde{Y}_S(c_j) \Gamma_j \right]
$$

where $W_i = \sum_{k=1}^{i} \omega_k$ and $\Gamma_j = \sum_{k=1}^{j} \gamma_k$. In order to relax this constraint, it is best to choose $U(v_1) = \Pi(c_m) = 0$ and $\tilde{Y}_S(c_j) = Y_S(c_{j+1})$ (recall that Y_S is decreasing and that $Y_S(c_j) \geq \tilde{Y}_S(c_j) \geq Y_S(c_{j+1})$) as well as $\tilde{Y}_B(v_i) = Y_B(v_i)$ (recall that Y_B is increasing and that $Y_B(v_{i+1}) \geq \tilde{Y}_B(v_i) \geq Y_B(v_i)$). Note that none of these variables is part of the objective (1) and therefore these choices are indeed optimal. With these choices the budget balance constraint can be written as

$$
\sum_{i=1}^{n} [\omega_i Y_B(v_i) v_i - (v_{i+1} - v_i) Y_B(v_i)(1 - W_i)] \geq \sum_{j=1}^{m} [\gamma_j Y_S(c_j) c_j + (c_{j+1} - c_j) Y_S(c_{j+1}) \Gamma_j] \tag{6}
$$

which is equivalent to

$$
\sum_{i=1}^{n} \sum_{j=1}^{m} y(v_i, c_j) \omega_i \gamma_j \left[v_i - (v_{i+1} - v_i) \frac{1 - W_i}{\omega_i} - c_j - (c_j - c_{j-1}) \frac{\Gamma_{j-1}}{\gamma_j} \right] \geq 0. \tag{7}
$$

Neglecting the monotonicity constraints on Y_S and Y_B for now, the mechanism design problem becomes maximizing (1) subject to (7). Hence, the optimal decision rule y must maximize the Lagrangian

$$
\mathcal{L}(y) = \sum_{i=1}^{n} \sum_{j=1}^{m} y(v_i, c_j) \omega_i \gamma_j \left[(1+\lambda) v_i - \lambda (v_{i+1} - v_i) \frac{1 - W_i}{\omega_i} - (1+\lambda) c_j - \lambda (c_j - c_{j-1}) \frac{\Gamma_{j-1}}{\gamma_j} \right] \tag{8}
$$

where $\lambda \geq 0$ is the Lagrange parameter of the budget balance constraint. As the Lagrangian is linear in y, the optimal decision rule is

$$
y^*(v_i, c_j) \begin{cases} = 1 & \text{if } v_i - (v_{i+1} - v_i) \frac{\lambda}{1+\lambda} \frac{1-W_i}{\omega_i} > c_j + (c_j - c_{j-1}) \frac{\lambda}{1+\lambda} \frac{\Gamma_{j-1}}{\gamma_j} \\ \in [0,1] & \text{if } v_i - (v_{i+1} - v_i) \frac{\lambda}{1+\lambda} \frac{1-W_i}{\omega_i} = c_j + (c_j - c_{j-1}) \frac{\lambda}{1+\lambda} \frac{\Gamma_{j-1}}{\gamma_j} \\ = 0 & \text{else.} \end{cases} \tag{9}
$$

This leaves us with two questions: First, is it possible that the budget balance constraint is non-

[3] Here I use the notational convention that $\sum_{k=j}^{0} \cdots = 0$ for any $j = 1, 2 \ldots$.

[4] Define $v_{n+1} = v_n$, $c_0 = c_1$ and $c_{m+1} = c_m$ for notational convenience and similarly $\tilde{Y}_b(v_{n+1}) = \tilde{Y}_S(c_{m+1}) = 0$.

binding? Second, will y^* satisfy the neglected monotonicity conditions? Since the signal distribution will be chosen in order to maximize expected welfare, it is unclear whether the usual monotone hazard rate conditions apply to W and Γ. In the following, it will be shown that it is never optimal to choose the information structure such that the budget balance constraint is slack (in this case information is too coarse) or such that the monotonicity constraint is binding (in this case information is too fine).

4. Optimal information structure

For most of this section I take the number of signals as given. That is, it is assumed that the support of F contains no more than n signals and the support of G contains no more than m signals.[5] This restriction is useful for several reasons. First, it simplifies notation and expostion. Second, two results will be shown at a later point in the paper that emphasize the relevance of finite information structures. More precisely, finite information structures turn out to be optimal if the true type distributions H_B and H_S have finite support. Even if this not the case, it will be shown that finite information structures achieve welfare levels arbitrarily close to maximal welfare. In the following, n and m will be assumed to be at least two. The justification for this is the following lemma which establishes that pooling all types on one signal is never welfare optimal.

Lemma 2. *The support of the signal distribution in the welfare optimal information structure contains at least two elements for each player.*

Given the restriction to no more than n (m) buyer (seller) signals, properties of the optimal information structure are characterized. I will show four main properties of the optimal information structure and mechanism: decision monotonicity, binding budget balance constraint, monotone partition structure and deterministic mechanism.

Decision monotonicity refers to the monotonicity conditions for incentive compatibility given in the previous section: Y_S has to be decreasing and Y_B increasing. Note that $\lambda = 0$ in (9) would imply that y^* is the first best rule and clearly this leads to monotone Y_S and Y_B (though not necessarily strictly monotone). In order to verify that neglecting the monotonicity constraints for Y_B and Y_S in the derivation of (9) was immaterial provided that the signal structure is optimal, it is therefore sufficient to concentrate on the case $\lambda > 0$; that is, the case where the budget balance constraint (7) binds. Suppose now the buyer's monotonicity constraint was binding, that is $Y_B(v_i) = Y_B(v_{i+1})$ for some $i \in \{1, \ldots, n-1\}$. The proof of the following lemma shows that "merging the two signals into one" would not affect the objective but strictly relax the binding budget balance constraint – a contradicition. The intuition is that merging the signals leads to coarser information and therefore to lower information rents. However, there is no downside in terms of welfare as $Y_B(v_i) = Y_B(v_{i+1})$ implies that the additional information present in the original information structure is not used to determine the efficient allocation. The lemma is stated and proven for a finite number of signals. However, this is for notational convenience only and the result holds generally as "merging signals" for which the monotonicity constraint binds generally relaxes the budget balance constraint without affecting welfare.

Lemma 3. *If the budget balance condition binds in the welfare optimal information structure with at most n buyer signals and m seller signals, then Y_S is strictly decreasing and Y_B is strictly increasing in the optimal mechanism.*

[5] For notational convenience, I will then state and prove the results assuming that there are n (m) buyer (seller) signals and all these signals have strictly positive probability, i.e. $\omega_i > 0$ and $\gamma_j > 0$ for all i and j. If it is optimal to use only $n^* < n$ ($m^* < m$) signals though n (m) signals are allowed, the results obviously still hold as the solution is equivalent to the solution with n^* (m^*) in place of n (m).

Lemma 3 establishes that Y_B and Y_S are strictly monotone if the budget balance constraint binds. The following lemm establishes that the budget balance will indeed hold with equality in the optima information structure. Intuitively, finer information structures and therefore a more efficient allocation would be feasible if the budget balance constraint was slack.

Lemma 4. *Assume that a fully informative signal structure does not achieve first best welfare. The budget balance constraint is then satisfied with equality in the welfare optimal information structure with at most n (m) buyer (seller) signals.*

The previous two lemmas established that the optimal decision rule is indeed characterized by (9) and neglecting the monotonicity constraints in its derivation is immaterial as Y_B and Y_S will be strictly monotone. It is now worthwhile to return to (9). This optimality condition can be stated in terms of virtual valuations. That is, a buyer with signal v_i trades with a seller of signal c_j if his virtual valuaton exceeds the one of the seller. The virtual valuation are defined as

$$\begin{aligned} VV_B(v_i) &= v_i - (v_{i+1} - v_i)\frac{\lambda}{1+\lambda}\frac{1-W_i}{w_i} \\ VV_S(v_j) &= c_j + (c_j - c_{j-1})\frac{\lambda}{1+\lambda}\frac{\Gamma_{j-1}}{\gamma_j}. \end{aligned}$$

Strict monotonicity of Y_B implies that higher buyer signals must lead to higher virtual valuations. It also implies that between the virtual valuation of any two buyer signals there has to be the virtual valuation of a seller signal. The reason is that otherwise the two buyer signals would have the same probability of trade, i.e. the monotonicity constraint holds with equality and it would be better to merge the signals. Hence, seller and buyer signals will alternate in terms of virtual valuations.

The first main result establishes that the optimal information structure is a monotone partition of the type space. That is, each signal v_i corresponds to an interval of types. If H_B is discrete, a monotone partition can assign a given type with positive probability mass with some probability to v_i and some probability to v_{i+1}. It is then easier to think of a partition of $[0, 1]$ where each signal v_i corresponds to an interval $(a_i, b_i] \subset [0, 1]$ such that (i) signal v_i has probability mass $b_i - a_i$ in F and (ii) the types in $H_B^{-1}((a_i, b_i])$ receive signal v_i where H_B^{-1} is the generalized inverse of H_B. The optimal information structure with no more than n elements can therefore be completely described by $n - 1$ cutoffs.

Proposition 1. *The optimal information structure with (at most) n buyer and m seller signals is a monotone partition (up to a measure zero set).*

The main idea behind the proof of proposition 1 is that an information structure that is not a monotone partition allows for both "mixing" and "demixing". Mixing refers here to the process of making an information structure coarser by moving two signals closer together. That is, if F assigns probabilities ω_i to v_i and ω_{i+1} to v_{i+1}, there is always a feasible information structure that uses the same probabilities but uses signals v'_i and v'_{i+1} a bit closer together. This can be achieved by sending the types that receive signal v_i (v_{i+1}) under F with some small probability the signal v'_{i+1} (v'_i) instead. If F is not a monotone partition, the opposite is possible: There is a feasible information structure that differs from F only by moving the signals v_i and v_{i+1} apart from one another. The proof shows that welfare can always be improved by one of the two operations if both, mixing and demixing, are possible. It follows that the optimal information structure has to be a monotone partition where further demixing is impossible.

The following lemma derives an additional result for discrete type distributions. The lemma complements proposition 1 well and the combination of these two results allows to establish a finite upper bound on the size of the support of F (G) in case the type distribution H_B (H_S) is finite.

Lemma 5. *Let the true type distribution of buyer valuations H_B be discrete and let its support be $\{\hat{v}_1, \hat{v}_2, \dots\}$. If $\hat{v}_i$ and $\hat{v}_{i+1}$ are in the support of the optimal signal distribution, then the optimal information structure assigns zero probability to all signals in $(\hat{v}_i, \hat{v}_{i+1})$. (An analogous result holds for sellers.)*

Proposition 1 and lemma 5 taken together imply that finite type distributions lead to finite welfare optimal information structures: A monotone partition of a finite distribution with k elements in its support could lead to a signal structure with at most $2k-1$ elements. With the restriction imposed by lemma 5, however, the optimal information partition can have at most k elements.[6]

Corollary 1. *Let the number of elements in the support of H_B (H_S) be finite and denote it by k. Then the support of the optimal signal structure for the buyer (seller) contains at most k elements.*

Corollary 1 is important because arbitrary distributions of types H_S and H_B can be approximated arbitrarily closely by a probability distribution with finite support. For these distributions, the optimal signal structure is finite by corollary 1 and a monotone partition of the type space by proposition 1. Consequently, the optimal information structure can in general be approximated arbitrarily closely by a finite monotone partition of the type space. This property provides some justification for the focus on optimal finite information structures in the preceding lemmas and proposition. The following lemma formalizes the just mentioned approximation idea.

Lemma 6. *Take any information structure (F, G) and denote expected welfare in this information structure (using the optimal mechanism) by W_{FG}. Then for any $\varepsilon > 0$ there exists an information structure (F_n, G_n) with finite support such that welfare under (F_n, G_n) (using the optimal mechanism) is at least $W_{FG} - \varepsilon$.*

The previous results established properties of the optimal information structure. The following result describes the optimal mechanism given the optimal information structure. In particular, it establishes that the optimal mechanism is deterministic. Note that the optimal mechanism for generic discrete information structures is not deterministic because generically the budget balance constraint does not hold with equality in deterministic mechanisms. The example in section 1.1 gives an illustration of this and also clarifies why stochastic mechanisms are not welfare maximizing: The only reason for a stochastic mechanism is to relax the binding incentive compatibility constraint for some signal and thereby the budget balance constraint. However, the example demonstrates that information design – in particular mixing the signal whose incentive compatibility constraint has to be relaxed with a worse signal – is a more efficient way of achieving the goal of relaxing these constraints.

Proposition 2. *Assume that a fully informative signal structure does not achieve first best welfare. Given the optimal information structure with n (m) buyer (seller) signals, $y(v_i, c_j) \in \{0, 1\}$ for all $i = 1, \dots, n$ and $j = 1, \dots, m$ in the optimal mechanism.*

5. Optimal binary signal structure

One important implication of corollary 1 for the binary case is that the optimal information structure has binary support if the true type distribution is binary. As binary type distributions do not only

[6] For example, a distribution with probability $1/2$ on 0 and probability $1/2$ on 1 could be monotonically partitioned into a signal structure putting probability $1/3$ on each element of $\{0, 1/2, 1\}$. However, lemma 5 does not allow a signal in $(0, 1)$ if both 0 and 1 have positive probability in the signal structure. Consequently, the support of the optimal signal structure will contain only two elements in this example.

provide more structure but are also often used in the (applied) literature (Kamenica and Gentzkow, 2011; Taneva, 2018), it makes sense to investigate the binary case in more detail.

Before exploiting the binary structure of the true binary distribution, two results are stated that make only use of the restrction $n = m = 2$. In other words, lemmas 7 and 8 also hold if only the signal distribution is restricted to be binary while the type distribution may not be binary.

One result generalizing the initial example in section 1.1 is that the optimal mechanism enforces trade if and only if the expected value is above expected cost. For simplicity, the signals concerning costs (value) are denoted in this secition by c_l (v_l) and c_h (v_h) with $c_h > c_l$ ($v_h > v_l$).

Lemma 7. *Consider the optimal mechanism under the optimal information structure for* $n = m = 2$*. Then* $y(v_l, c_h) = 0$ *and* $y(v_h, c_l) = 1$*.*

Intuitively, 7 has to be true as $y(v_l, c_h) > 0$ would imply that $v_l \geq c_h$ and therefore trade irrespective of the signal at price $(v_l + c_h)/2$ would be optimal. This is outcome equivalent to pooling all types and cannot be optimal as information design could be used to rule out some inefficient trades.

Lemma 8. *Consider the optimal mechanism under the optimal information structure for* $n = m = 2$*. Then the optimal mechanism enforces trade if and only if the buyer signal exceeds the seller signal.*

Clearly, the "only if" part holds by (9). To illustrate the "if" part of lemma 8 consider signals c_h and v_l. By lemma 7, $y(v_l, c_h) = 0$ and therefore it is necessary to establish that $c_h \geq v_l$. If this was not the case, a fixed price mechanism at price $(v_l + c_h)/2$ (and trade irrespective of signal) would clearly be feasible and improve welfare. Put differently, if trade did not take place although expected value is above expected cost (conditional on signals), then a fixed price mechanism would improve welfare. This method of proof is admittedly specific to the binary signal case.

5.1. Binary type distribution

This section considers the case where the true type distribution is binary. For future reference, corollary 1 is restated in a slightly more precise fashion for the binary type case.

Corollary 2. *Let the true buyer valuation (seller cost) distribution have binary support. Then the optimal signal structure for the buyer (seller) has binary support and at least one element of the support is also an element of the support of the true valuation (cost) distribution.*

By corollary 2, the optimal signal distribution is binary if types are binary and one of the valuation signals as well as one of the cost signals must be fully informative. At some places it will be useful to denote the true cost and valuation types by $\underline{c} < \bar{c}$ and $\underline{v} < \bar{v}$. Assumption 1 can then be written as $\underline{c} < \underline{v} < \bar{c} < \bar{v}$.

Lemma 9. *Consider the optimal mechanism under the optimal information structure for binary type support. Then* $y(v_l, c_l) = 1 = y(v_h, c_h)$ *and both* $v_h = \bar{v}$ *and* $c_l = \underline{c}$*.*

Lemma 9 leaves only the option that trade occurs unless both signals are "bad". The lemma does not entirely describe the information structure as it does not indicate with which probability $\bar{v}$ ($\underline{c}$) types receive the v_l (c_h) signal. However, the budget balance constraint has to be satisfied with equality (unless a fully informative signal structure yields a budget surplus). Hence, the search for the optimal information structure is equivalent to a maximization problem over two variables with one constraint or – as the constraint can be solved explicitly for one of the variables – an optimization problem over one variable without constraint, see appendix D. It is even possible to show that the objective in the latter problem is convex and therefore the solution is a corner solution. This means that one of the two players will have a fully informative signal while the other's signal has just enough noise to ensure that budget balance holds. The optimal information structure is therefore one of the following two[7]

1. *buyer revealing*: $v_h = \bar{v}$, $v_l = \underline{v}$, $c_l = \underline{c}$ and $c_h = \frac{\gamma - \gamma_l^{BB}(\bar{\omega})}{1-\gamma_l^{BB}(\bar{\omega})}\underline{c} + \frac{1-\gamma}{1-\gamma_l^{BB}(\bar{\omega})}\bar{c}$ while $\gamma_l = \gamma_l^{BB}(\bar{\omega})$ and $\omega_h = \bar{\omega}$

2. *seller revealing*: $v_h = \bar{v}$, $v_l = \frac{\bar{\omega} - \omega_h^{BB}(\underline{\gamma})}{1-\omega_h^{BB}(\underline{\gamma})}\bar{v} + \frac{1-\bar{\omega}}{1-\omega_h^{BB}(\underline{\gamma})}\underline{v}$, $c_l = \underline{c}$, $c_h = \bar{c}$ while $\gamma_l = \underline{\gamma}$ and $\omega_h = \omega_h^{BB}(\underline{\gamma})$.

It is straightforward to compute welfare in each of the two solution candidates above and the candidate achieving the highest welfare is the optimal information structure. This comparison leads to the following result that completely describes the optimal information structure and mechanism in case of binary types.

Proposition 3. *Let the support of H_S and H_B be binary. Then the optimal information structure is* buyer revealing *if and only if*

$$\frac{(1-\underline{\gamma})(\bar{v}-\bar{c})}{(1-\bar{\omega})(\underline{v}-\underline{c})}\left(\bar{\omega} - \frac{1}{2}\left(1 + \frac{\underline{\gamma}(\bar{v}-\underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{\underline{\gamma}(\bar{v}-\underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}\right)^2 - \frac{\underline{\gamma}\bar{\omega}(\bar{v}-\underline{v}) + \underline{\gamma}(\underline{v}-\underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}}\right)$$
$$\geq \left(\underline{\gamma} - \frac{1}{2}\left(1 + \frac{\bar{\omega}(\bar{v}-\underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{\bar{\omega}(\bar{v}-\underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}\right)^2 - \frac{\bar{\omega}(\bar{v}-\bar{c}) + \bar{\omega}\underline{\gamma}(\bar{c}-\underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}}\right)$$

and seller revealing *if the reverse inequality holds.*

The resulting welfare can be compared to first best welfare

$$W^{fb} = \bar{\omega}\bar{v} + (1-\bar{\omega}) * \underline{\gamma}\underline{v} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{\omega}\bar{c}.$$

Note that it is without loss of generality to set $\bar{v} = 1$: Dividing all types by $\bar{v}$ will divide all constraints as well as the objective by $\bar{v}$ and therefore not affect the optimization problem. (Put differently, signals in the optimal information structure will be the previous optimal signals divided by $\bar{v}$. First and second best welfare will be divided by $\bar{v}$ as well.) With this normalization each of the remaining parameters, i.e. $\underline{v}$, $\underline{c}$, $\bar{c}$, γ_l, ω_h, is in the compact set $[0, 1]$ and consequently it is easy to numerically search for the parameter constellation in which the ratio of second best and first best welfare is minimal. Note that the just described normalization does not affect the ratio of second best and first best welfare. I computed this ratio numerically for all parameter values on a grid with stepsize 0.01, i.e. all parameter values $\bar{\omega}, \underline{\gamma} \in \{0.01, 0.02, \ldots, 0.99\}$ and $\underline{c} \in \{0.0, 0.01, \ldots, 0.97\}$, $\underline{v} \in \{\underline{c}+0.01, \ldots, 0.98\}$, $\bar{c} \in \{\underline{v}+0.01, \ldots, 0.99\}$ are considered. The lowest ratio was 0.95417 which was achieved at $\bar{\omega} = \underline{\gamma} = 0.04$, $\underline{c} = 0$, $\bar{c} = 0.99$, $\underline{v} = 0.01$. This means that the combination of information and mechanism design can limit the loss due to asymmetric information to less than 5% in a binary type bilateral trade setting. The ratio of first best to second best welfare when using the optimal mechanism but not using information design is a natural comparison point. In this case the lowest welfare ratio equals 0.89189 which was achieved at the same parameter constellation, i.e. $\bar{\omega} = \underline{\gamma} = 0.04$, $\underline{c} = 0$, $\bar{c} = 0.99$, $\underline{v} = 0.01$. This shows that information design can close more than half of the welfare gap left by mechanism design in a binary type bilateral trade setting.

[7] The function $\gamma_l^{BB}(\omega_h)$, which is defined in appendix D, gives the γ_l necessary to satisfy budget balance with equality for a given ω_h. $\omega_h^{BB}(\gamma_l)$ is defined analogously.

6. Example: Uniform type distribution

Another often used example is the uniform distribution on $[0, 1]$, i.e. $H_S(x) = H_B(x) = x$ for $x \in [0, 1]$. In this setting, first best welfare equals $1/6 = 0.1\bar{6}$. With fully revealing signals the optimal mechanism is to trade if and only if $v - c \geq 1/4$ and this leads to second best welfare equal to $9/64 = 0.140625$. Without information design, only 84.375% of first best welfare can be achieved due to asymmetric information. Table 3 presents results of a numerical analysis of this problem in which the optimal information partition with n buyer and n seller signals was derived.[8] For $n \geq 5$, the optimization algorithm used less than n types and welfare did not increase further. Information design closes almost the whole gap to first best welfare in this example as 97.55% of first best welfare can be achieved in the optimal information structure.

n	welfare	W/W^{fb}	buyer signals	seller signals
1	0	0	0.5	0.5
2	0.1482	.8892	0.1666, 0.6666	0.3333, 0.8333
3	0.16	.9667	0.1, 0.4, 0.8	0.2, 0.6, 0.9
4	0.1625	.9753	0.05030, 0.2482, 0.5474, 0.8494	0.1018, 0.3536, 0.6765, 0.9247
5	0.1626	.9755	0.0525, 0.2322, 0.5356, 0.8406, 0.9847	0.0855, 0.3386, 0.6660, 0.8919, 0.9790

Table 3: Optimal information structures for uniform type distributions on $[0, 1]$ (numerical analysis, rounded to fourth digit)

7. Conclusion

This paper derived the welfare optimal information structure and mechanism in a bilateral trade setting if true types are binary. While the derivation is not straightforward the resulting information structure and mechanism are strikingly simple. The optimal information structure is fully informative for one player and binary for the other player. The latter player receives either a signal fully revealing that he is a "good type" or a noisy signal. The optimal information structure renders the use of complicated mechanisms unnecessary: The optimal mechanism is deterministic and enforces trade if and only if – conditional on the signals – expected value is above expected costs.

Welfare in the optimal information structure can be interpreted as an upper bound on the welfare achievable in light of asymmetric information by any institutional framework. Consequently, the welfare loss compared to first best can be interpreted as the welfare loss that is fully attributable to information asymmetries. In the binary type setting this information loss is less than 5% of first best welfare. This is significantly less than the welfare loss without information design (while using the welfare optimal mechanism as in Myerson and Satterthwaite (1983)) which can exceed 10%.

With more general finite type distributions, the optimal information structure is a monotone partition of the type space and the optimal mechanism is deterministic. For type distributions with infinite support, welfare under the optimal information structure can be approximated arbitrarily closely by welfare in finite information structures that are monotone partitions of the type space. Generally, the budget balance constraint will hold with equality while the monotonicity constraint will be slack. If the type distribution is finite, the number of signals will never exceed the number of types.

[8]The numerical analysis first creates a grid of possible information structures with n buyer and seller signals and maximizes welfare by brute force on this grid. The optimal information structure from this brute force method is used as a starting value for an optimization algorithm. The algorithm used is "MMA (Method of Moving Asymptotes)" as implemented in the NLopt package, see Johnson (2019) and Svanberg (2002). The code is available on the website of the author (`https://schottmueller.github.io/`). The usual disclaimer for numerical work applies, i.e. the solution is not exact and information partitions leading to even higher welfare are in principle possible.

Appendix

A. Correlated signals

In the bilateral trade setup of this paper it is straightforward to show that first best welfare is achievable if one considers correlated information structures. To this end, consider a signal structure that maps each pair of types (v, c) to itself, i.e. each player receives a signal equal to the true type vector (v, c). Amend this signal structure with the mechanism

$$
\begin{aligned}
y((v_B, c_B), (v_S, c_S)) &= \begin{cases} 1 & \text{if } v_B = v_S \geq c_B = c_S \\ 0 & \text{else} \end{cases} \\
t_B((v_B, c_B), (v_S, c_S)) = t_S((v_B, c_B), (v_S, c_S)) &= \begin{cases} (v_B + c_B)/2 & \text{if } v_B = v_S \geq c_B = c_S \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

where the first (second) argument in y, t_B and t_S is the reported buyer (seller) signal. It is straighforward to see that this mecahnism is incentive compatible and satisfies the participation constraint. Most importantly, it achieves first best welfare by essentially eliminating the information asymmetry between buyer and seller.

B. Auxilliary results

Lemma 10. *Take a direct mechanism (y, t_S, t_B) that satisfies* (IR)*,* (IC$_\mathrm{B}$)*,* (IC$_\mathrm{S}$) *and ex ante budget balance. Then there is a direct mechanism $(y, \tilde{t}_S, \tilde{t}_B)$ that satisfies* (IR)*,* (IC$_\mathrm{B}$)*,* (IC$_\mathrm{S}$) *and the ex post budget balance constraint $\tilde{t}_S(v_i, c_j) = \tilde{t}_B(v_i, c_j)$ for all $v_i \in \{v_1, \dots, v_n\}$ and $c_j \in \{c_1, \dots, c_m\}$.*

Proof. With a slight abuse of notation denote by $T_S(v_i) = \sum_{j=1}^m t_s(v_i, c_j)\gamma_j$ the expected transfer of the seller conditional on the buyer type being v_i. Now define the new payment rules

$$
\begin{aligned}
\tilde{t}_B(v_i, c_j) &= t_B(v_i, c_j) + [t_S(v_i, c_j) - t_B(v_i, c_j)] - [T_S(v_i) - T_B(v_i)] \\
\tilde{t}_S(v_i, c_j) &= t_s(v_i, c_j) - [T_S(v_i) - T_B(v_i)] .
\end{aligned}
$$

Clearly, $\tilde{t}_S(v_i, c_j) = \tilde{t}_B(v_i, c_j)$ and therefore ex post budget balance holds. Furthermore, $\tilde{T}_B(v_i) = T_B(v_i)$ for all v_i and similarly $\tilde{T}(c_j) = T(c_j)$ for all c_j by the assumption that (y, t_S, t_B) is ex ante budget balanced. As y – and therefore Y_s and Y_B – di not change, this implies that $(y, \tilde{t}_S, \tilde{t}_B)$ staisfies (IR), (IC$_\mathrm{B}$), (IC$_\mathrm{S}$) because (y, t_S, t_B) did. □

C. Proofs of results in the text

Proof of lemma 1: *If:* Let (4) hold and Y_B be increasing. Take $i > k$. Iterating (4), yields

$$
U(v_i) = U(v_k) + \sum_{j=k}^{i-1} \tilde{Y}_B(v_j)(v_{j+1} - v_j). \tag{10}
$$

As $\tilde{Y}_B(v_j) \geq Y_B(v_j)$ and Y_B is increasing, this implies

$$
\begin{aligned}
U(v_i) &\geq U(v_k) + \sum_{j=k}^{i-1} Y_B(v_k)(v_{j+1} - v_j) \\
&= U(v_k) + Y_B(v_k)(v_i - v_j).
\end{aligned}
$$

Hence, ($\mathrm{IC_B}$) is satisfied for v_i and v_k. Similarly starting from (10), $\tilde{Y}_B(v_j) \leq \tilde{Y}_B(v_{j+1})$ and Y_B being increasing implies

$$U(v_i) \leq U(v_k) + \sum_{j=k}^{i-1} Y_B(v_i)(v_{j+1} - v_j)$$

and therefore $U(v_k) \geq U(v_i) + Y_B(v_i)(v_k - v_i)$ which means that ($\mathrm{IC_B}$) is satisfied for v_k and v_i.

Only if: Let ($\mathrm{IC_B}$) being satisfied. For $k = i - 1$, ($\mathrm{IC_B}$) is equivalent to $U(v_i) - U(v_{i-1}) \geq Y_B(v_{i-1})(v_i - v_{i-1})$. Using the incentive constraint that v_{i-1} does not want to misrepresent as v_i, ($\mathrm{IC_B}$) can be rearranged to $U(v_{i-1}) - U(v_i) \geq Y_B(v_i)(v_{i-1} - v_i)$. Taking these two inequalities together gives

$$Y_B(v_i) \geq \frac{U(v_i) - U(v_{i-1})}{v_i - v_{i-1}} \geq Y_B(v_{i-1}).$$

Hence, Y_B is increasing and (4) holds with $\tilde{Y}_B(v_{i-1}) = [U(v_i) - U(v_{i-1})]/[v_i - v_{i-1}]$.

The proof for the seller is analogous. □

Proof of lemma 2: Suppose to the contrary that all seller types are pooled on one signal $\mathbb{E}[c]$. In this case, the optimal mechanism is clearly a fixed price mechanism with price equal to $\mathbb{E}[c]$. Consequently, the optimal information structure for the buyer is without loss of generality binary: One signal v_l for all types below $\mathbb{E}[c]$ and one signal v_h for all types above $\mathbb{E}[c]$. By assumption 1, v_h has positive probability mass denoted by ω_h. The argument now depends on whether v_l has positive probability mass.

As a first case assume that v_l has positive probability. I will change now the seller information structure and the mechanism in two steps and show that a welfare increasing budget balanced improvement exists. In the first step, change the information structure of the seller to an information structure with two signals $c_l = v_l$ and $c_h \in (\mathbb{E}[c], v_h)$ while maintaining the mechanism $y(v_h, \cdot) = 1$ and $y(v_l, \cdot) = 0$. By assumption 1, such an information structure in which both c_l and c_h have positive probability exists.[9] Note that welfare is the same as before because the trading probabiltiy between any two types have not changed. Furthermore, the budget balance constraint can be written as $\omega_h(v_h - c_h) > 0$, i.e. the budget balance condition is slack. In a second step, increase $y(v_l, c_l)$ from 0 to $\varepsilon > 0$ where ε is chosen small enough to keep the budget balance condition, which reads $\omega_h(v_h - c_h) - \varepsilon\gamma_l(\omega_h(v_h - c_l) - v_l + c_l) \geq 0$, slack. As $v_l = c_l$, welfare is again unchanged. In a final step, change the seller's information structure such that γ_l, the probabiltiy of receivng the low signal, stays the same but $c_l = v_l - \varepsilon'$ and $c_h \in (\mathbb{E}[c], v_h)$ which is again possible by assumption 1 for $\varepsilon' > 0$ small enough. As $y(v_l, c_l) = \varepsilon \in (0, 1)$, this increases expected welfare. For $\varepsilon' > 0$ small enough the budget balance constraint is not violated as it is continuous in ε' and was slack for $\varepsilon' = 0$. This establishes an information structure and mechanism satisfying budget balance and yielding a strictly higher welfare than the initial structure in which the seller's types were pooled.

As second case assume that v_l has zero probability mass, i.e. $\mathbb{E}[c] \leq \min supp(H_B)$ and both seller and buyer types are pooled on a single signal each in the supposedly optimal information structure.[10] The optimal mechanism clearly enforces trade with probability 1 in this case. I will change the signal structures in several steps maintaining budget balance in each step and (weakly) increasing welfare in each step. By assumption 1, there exists an $s \in [\min supp(H_S), \max supp(H_B)]$ and an $\varepsilon > 0$ such that $H_B(s) > \varepsilon$ and $1 - H_S(s) > \varepsilon$. In a first step, change both players information structure to binary signals such that signal $c_h = v_l = s$ is sent with probability ε and the signals $c_l < c_h$ and $v_h > v_l$ are sent with probability $1 - \varepsilon$ (where c_l and v_h are chosen such that the expected value equals the expected value of the type distribution; by the definition of s and ε such a distribution is feasible). The

[9] For example, let $\gamma_l = H_S(v_l)/2$ where $H_S(v_l) > 0$ by assumption 1 as $v_l \geq \min supp(H_B) > \min supp(H_S)$.

[10] An argument analogous to the first case establishes also that $\mathbb{E}[v] \geq \max supp(H_S)$ in this case.

new information structure leads to the same welfare when maintaining trade with probability 1 and is clearly budget balanced as a fixed price mechanism with price s is possible. In a second step, change the mechanism by setting $y(v_l, c_h) = 0$. As $v_l = c_h$, this does not affect welfare and as the change relaxes the budget balance constraint, this constraint holds now with strict inequality. In a final step, increase c_h slightly and decrease c_l slightly while both signals are still sent with probabilities ε and $1-\varepsilon$ (the decrease in c_l is, of course, chosen such that the expected value is maintained). This is feasible as $1 - H_S(c_h) = 1 - H_S(s) > \varepsilon$ by the definition of ε. Since the budget balance constraint is continuous in signals, a sufficiently small change will not violate this constraint. Furthermore, welfare is strictly increased as costs conditional on trade decreases – due to $Y_S(c_h) < Y_S(c_l)$ – and the probability of trade is unaffected. $\square$

Proof of lemma 3: Suppose to the contrary $Y_B(v_i) = Y_B(v_{i+1})$ for some $i \in \{1, \dots, n-1\}$ in the optimal mechanism under the optimal information structure. In case the monotonicity constraint binds for more than two signals, let v_i be the lowest signal for which it binds. Now consider an information structure in which signals v_i and v_{i+1} are merged, that is, every type v that got either signal v_i or v_{i+1} will now get signal

$$\tilde{v} = \frac{\omega_i}{\omega_i + \omega_{i+1}} v_i + \frac{\omega_{i+1}}{\omega_i + \omega_{i+1}} v_{i+1}$$

and nothing changes for other types. Adapt the decision rules y by letting

$$\tilde{y}(\tilde{v}, c_j) = \frac{\omega_i}{\omega_i + \omega_{i+1}} y(v_i, c_j) + \frac{\omega_{i+1}}{\omega_i + \omega_{i+1}} y(v_{i+1}, c_j).$$

Note that this construction implies that $\tilde{Y}_S = Y_S$ and $\tilde{Y}_B(v_k) = Y_B(v_k)$ for all $k \in \{1, \dots, i-1, i+2, \dots, n\}$ and in particular $\tilde{Y}_B(\tilde{v}) = Y_B(v_i) = Y_B(v_{i+1})$. The objective (1) which can be written as $\sum_i \omega_i Y_B(v_i) v_i - \sum_j \gamma_j Y_S(c_j) c_j$ is therefore unchanged by the merging of signals. However, constraint (7) is strictly relaxed by the merging of signals: Note that (7) can be written as

$$\left\{ \sum_{i=1}^{n} Y_B(v_i)\omega_i \left[v_i - (v_{i+1} - v_i)\frac{1 - W_i}{\omega_i} \right] \right\} - \left\{ \sum_{j=1}^{m} Y_S(c_j)\gamma_j \left[c_j + (c_{j+1} - c_j)\frac{\Gamma_j}{\gamma_j} \right] \right\} \geq 0.$$

The merging of types affects only the two terms for v_i and v_{i+1} as Y_S and Y_B for other signals were not affected. Hence, the relevant two terms are (using the notation $\tilde{\omega} = \omega_i + \omega_{i+1}$)

$$\begin{aligned}
& -Y_B(v_{i-1})v_i(1 - W_{i-1}) + Y_B(v_i)\left[\omega_i v_i - (v_{i+1} - v_i)(1 - W_i)\right] + Y_B(v_{i+1})\left[\omega_{i+1} v_{i+1} - (v_{i+2} - v_{i+1})(1 - W_{i+1})\right] \\
= \; & -Y_B(v_{i-1})v_i(1 - W_{i-1}) + \tilde{Y}_B(\tilde{v})\left[\tilde{v}\tilde{\omega} - (v_{i+1} - v_i)(1 - W_i) - (v_{i+2} - v_{i+1})(1 - W_{i+1})\right] \\
= \; & -Y_B(v_{i-1})\tilde{v}(1 - W_{i-1}) + Y_B(v_{i-1})(\tilde{v} - v_i)(1 - W_{i-1}) \\
& + \tilde{Y}_B(\tilde{v})\left[\tilde{v}\tilde{\omega} - (v_{i+1} - v_i)(1 - W_{i+1}) - (v_{i+2} - v_{i+1})(1 - W_{i+1})\right] - \tilde{Y}_B(\tilde{v})(v_{i+1} - v_i)\omega_{i+1} \\
= \; & -Y_B(v_{i-1})\tilde{v}(1 - W_{i-1}) + Y_B(v_{i-1})(\tilde{v} - v_i)(1 - W_{i+1} + \omega_i + \omega_{i+1}) \\
& + \tilde{Y}_B(\tilde{v})\left[\tilde{v}\tilde{\omega} - (v_{i+2} - v_i)(1 - W_{i+1})\right] - \tilde{Y}_B(\tilde{v})(v_{i+1} - v_i)\omega_{i+1} \\
= \; & -Y_B(v_{i-1})\tilde{v}(1 - W_{i-1}) + Y_B(v_{i-1})(\tilde{v} - v_i)(1 - W_{i+1}) + Y_B(v_{i-1})(v_{i+1} - v_i)\omega_{i+1} \\
& + \tilde{Y}_B(\tilde{v})\left[\tilde{v}\tilde{\omega} - (v_{i+2} - \tilde{v})(1 - W_{i+1})\right] - \tilde{Y}_B(\tilde{v})(v_{i+1} - v_i)\omega_{i+1} - \tilde{Y}_B(\tilde{v})(\tilde{v} - v_i)(1 - W_{i+1}) \\
= \; & -Y_B(v_{i-1})\tilde{v}(1 - W_{i-1}) + \tilde{Y}_B(\tilde{v})\left[\tilde{v}\tilde{\omega} - (v_{i+2} - \tilde{v})(1 - W_{i+1})\right] \\
& + (Y_B(v_{i-1}) - \tilde{Y}_B(\tilde{v}))(\tilde{v} - v_i)(1 - W_{i+1}) + (Y_B(v_{i-1}) - \tilde{Y}_B(\tilde{v}))(v_{i+1} - v_i)\omega_{i+1} \\
< \; & -Y_B(v_{i-1})\tilde{v}(1 - W_{i-1}) + \tilde{Y}_B(\tilde{v})\left[\tilde{v}\tilde{\omega} - (v_{i+2} - \tilde{v})(1 - W_{i+1})\right]
\end{aligned}$$

where the first equality uses $Y_B(v_i) = Y_B(v_{i+1}) = \tilde{Y}_B(\tilde{v})$ and the definition of $\tilde{v}$, the inequality uses $\tilde{Y}_B(\tilde{v}) = Y_B(v_i) > Y(v_{i-1})$ (recall that i was the lowest pooled type). Note that the term we end up with is exactly the term refering to $\tilde{v}$ in (7) under the modified $\tilde{y}$. Consequently, the merging of signals strictly relaxed the binding constraint without affecting the objective which contradicts the optimality of y.

The proof for the seller is analogous. □

Proof of lemma 4: Suppose the budget balance constraint was slack under the optimal signal structure (F, G), i.e. held with strict inequality, but violated under the truthful information structure (H_B, H_S). In this case $\lambda = 0$ under (F, G) and (9) implies that trade takes place if and only if value is above costs. By lemma 3, the optimal information structure can be chosen such that Y_B or Y_S are strictly monotone. (If these functions are flat, merging types will relax the budget balance constraint without affecting welfare, see the proof of lemma 3.) It is therefore without loss of generality to assume that Y_S and Y_B are strictly monotone. Note that welfare can be written as $\sum_{i=1}^{n} \omega_i v_i Y_B(v_i) - \sum_{j=1}^{m} \gamma_j c_j Y_S(c_j)$. By the assumption that the budget balance constraint is violated in case of a fully informative signal to both players, at least one player's signal is noisy. For concreteness, say this is the buyer, i.e. $F \neq H_B$ and H_B is a mean preserving spread of F. The intuition is now as follows: Consider a feasible $\tilde{F}$ that is close to F but somewhat more spread out than F. As $\tilde{F}$ is feasible, $\sum_{i=1}^{n} \omega_i v_i$ is the same under F and $\tilde{F}$ but as $\tilde{F}$ is more spread out $\sum_{i=1}^{n} \omega_i v_i Y_B(v_i)$ is higher under $\tilde{F}$ than under F because Y_B is strictly increasing. Choosing $\tilde{F}$ close to F, the budget balance constraint is not violated under $\tilde{F}$ and therefore (F, G) cannot be optimal. More concretely, I distinguish two cases.

First, consider the case that F is not the result of a monotone partition of the type space. Then there exist signals v_i and v_j with $v_i > v_j$ and $\varepsilon, \varepsilon' > 0$ such that reducing v_j to $v_j - \varepsilon$ and increasing v_i to $v_i + \varepsilon'$ while keeping all other signals and the probability of each signal the same is a feasible signal distribution. By feasibility $\sum_{i=1}^{n} \omega_i v_i$ did not change and therefore $\sum_{i=1}^{n} \omega_i v_i Y_B(v_i)$ is higher because Y_B is strictly increasing. This contradicts the optimality of F. Given that the budget balance constraint was slack initially, it will still hold for ε, $\varepsilon' > 0$ sufficiently small as (6) is continuous in v_i and v_j.

Second, consider the case that F is the result of a monotone partition of the type space. Take two signals v_i and v_{i+1} and change the signal structure by moving the $\varepsilon > 0$ highest types receiving signal v_i to signal v_{i+1} (i.e. reduce the partition cutoff between these two signals slightly). This will reduce both v_i and v_{i+1} as well as ω_i while increasing ω_{i+1}. Denote the new signals by $\tilde{v}_i$ and $\tilde{v}_{i+1}$. As $\omega_i v_i + \omega_{i+1} v_{i+1} = (\omega_i - \varepsilon)\tilde{v}_i + (\omega_{i+1} + \varepsilon)\tilde{v}_{i+1}$ by the feasibility of the changed signal distribution, $(\omega_i - \varepsilon)\tilde{v}_i < \omega_i v_i$ and $(\omega_{i+1} + \varepsilon)\tilde{v}_{i+1} > \omega_{i+1} v_{i+1}$. But this implies that $\sum_{i=1}^{n} \omega_i v_i Y_B(v_i)$ is higher under the changed information structure as Y_B is strictly increasing. For $\varepsilon > 0$ sufficiently small, the budget balance constraint is still satisfied given that it was initially slack as (6) is continuous in v_i, v_{i+1}, ω_i and ω_{i+1}. □

Proof of proposition 1: We show the result for the buyer. Suppose by way of contradiction that the optimal $(v_i)_{i=1}^{n}$ and $(\omega_i)_{i=1}^{n}$ do not form a monotone partition (up to a measure zero set). This implies that there exists some $i \in \{1, \ldots, n\}$ and a set of true valuation types N_i with some mass $\eta > 0$ that receives signal v_i and a set of true valuation types N_{i+1} with mass $\eta > 0$ that receives signal v_{i+1} such that $\mathbb{E}[v|v \in N_i] > \mathbb{E}[v|v \in N_{i+1}]$. We will return to these sets later.

Consider for now the optimization problem of maximizing expected welfare subject to budget balance: Maximize expected welfare, i.e. $\sum_i \sum_j \omega_i \gamma_j (v_i - c_j) y(v_i, c_j)$, over y, $(v_i)_{i=1}^{n}$, $(c_j)_{j=1}^{m}$, ω_i and γ_j subject to (7), i.e. the budget balance constraint. Let the domain for y be $[0, 1]$ and the domain for $(\omega_i)_{i=1}^{n}$, $(v_i)_{i=1}^{n}$ is the set of all distributions such that F is a mean preserving spread of these distributions. Respectively, the domain of $(\gamma_j)_{j=1}^m$, $(c_j)_{j=1}^m$ is such that G is a mean preserving spread of these distributions. Note that incentive compatibility and individual rationality will be automatically satisfied by the solution due to substituting the expressions from lemma 1 and individual rationality into the budget constraint in order to obtain (7). (By lemma 3 the monotonicity constraint is slack.) That is, the solution to this program will be the optimal information structure and mechanism if the number of buyer (seller) signals is restricted to no more than n (m).[11] Writing the Lagrangian for this optimization problem with Lagrange parameters λ for the budget balance constraint yields:

$$\mathcal{L} \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{m}\left\{y(v_i,c_j)\omega_i\gamma_j\left[(1+\lambda)v_i-\lambda(v_{i+1}-v_i)\frac{1-W_i}{\omega_i}-(1+\lambda)c_j-\lambda(c_j-c_{j-1})\frac{\Gamma_{j-1}}{\gamma_j}\right]\right\}$$

A solution to this finite-dimesional problem exists by the Weierstrass theorem as the feasible set is compact and non-empty and the objective is continuous. Consider $\mathcal{L}$ evaluated at the solution values for y, $(\omega_i)_{i=1}^n$, $(\gamma_j)_{j=1}^m$ and $(c_j)_{j=1}^m$. Given that, the optimal values for $(v_i)_{i=1}^n$ have to maximize $\mathcal{L}$ (within the feasible set of v_i, i.e. all those $(v_i)_{i=1}^n$ that yield together with $(\omega_i)_{i=1}^n$ a distribution such that F is a mean preserving spread of it). Now consider the following family of buyer valuation distributions indexed by ε which I denote by $(\tilde{v}_i)_{i=1}^n$: Fix all valuations apart from some $\tilde{v}_i$ and $\tilde{v}_{i+1}$ at their optimal levels (i.e. at the values that are part of the solution of the maximization problem above) and let

$$\begin{aligned}\tilde{v}_i(\varepsilon) &= \frac{(\omega_i-\varepsilon)v_i+\varepsilon v_{i+1}}{\omega_i}\\ \tilde{v}_{i+1}(\varepsilon) &= \frac{(\omega_{i+1}-\varepsilon)v_{i+1}+\varepsilon v_i}{\omega_{i+1}}\end{aligned}$$

where v_i and v_{i+1} are the solution values in the maximization problem above. As $v_i(0)=v_i$ and $v_{i+1}(0)=v_{i+1}$, the auxiliary maximization problem of maximizing $\mathcal{L}$ over ε (where all variables apart from $\tilde{v}_i$ and $\tilde{v}_{i+1}$ are fixed at their optimal solution) must be solved by $\varepsilon=0$ (if the information structure is feasible for ε in an open neighborhood around 0). The corresponding derivative of $\mathcal{L}$ with respect to ε is

$$\begin{aligned}\frac{d\mathcal{L}}{d\varepsilon} &= (v_{i+1}-v_i)\left[Y_B(v_i)(1+\lambda+\lambda(1-W_i)/\omega_i)-Y_B(v_{i-1})\lambda(1-W_{i-i})/\omega_i\right]\\ &\quad -(v_{i+1}-v_i)\left[Y_B(v_{i+1})(1+\lambda+\lambda(1-W_{i+1})/\omega_{i+1})-Y_B(v_i)\lambda(1-W_i)/\omega_{i+1}\right] \qquad (11)\end{aligned}$$

Note that the derivative does not depend on ε, i.e. $\mathcal{L}$ in the auxiliary maximization problem is linear in ε. It is straightforward to see that $(\tilde{v}_i)_{i=1}^n$ is feasible for $\varepsilon \geq 0$ if $\varepsilon \geq 0$ is not too high. (Essentially $\tilde{v}_i$ and $\tilde{v}_{i+1}$ use the optimal information structure which is feasible and then swap the signal for ε of those types receiving signals v_i and v_{i+1} in the optimal information structure. Clearly, this does not change ω_i or ω_{i+1} and yields a new feasible information structure.) I will now show that $(\tilde{v}_i)_{i=1}^n$ are also feasible for $\varepsilon < 0$ (not too far from 0) if the optimal information structure is not a monotone partition. After ruling out that the slope of $\mathcal{L}$ in ε is zero, this will complete the proof as feasibility for ε in an open interval around 0 means that $\varepsilon = 0$ cannot maximize the linear $\mathcal{L}$ in the auxiliary problem. This contradiction establishes that the optimal information structure (for the buyer) must be a monotone partition.

To see that $\varepsilon < 0$ is feasible, consider changing the information structure by swapping the signal of mass $\tau < \eta$ in N_i and N_{i+1}, i.e mass $\tau < \eta$ of the types in N_i receives signal v_{i+1} (instead of v_i) and

[11]Strictly speaking one should also add constraints enforcing $v_{i+1} - v_i \geq 0$ and $c_{j+1} \geq c_j$ which will, however, not change the argument below and only clutter notation further.

mass τ in N_{i+1} receives signal v_i (instead of v_{i+1}). This is clearly feasible and does not change ω_i or ω_{i+1} but the expected valuation when receiving signals v_i or v_{i+1} changes to

$$\begin{aligned}\tilde{v}_i(\tau) &= \frac{\omega_i v_i - \tau\left(\mathbb{E}[v|v \in N_i] - \mathbb{E}[v|v \in N_{i+1}]\right)}{\omega_i} \\ \tilde{v}_{i+1}(\tau) &= \frac{(\omega_{i+1} v_{i+1} + \tau\left(\mathbb{E}[v|v \in N_i] - \mathbb{E}[v|v \in N_{i+1}]\right)}{\omega_{i+1}}.\end{aligned}$$

Choosing $\tau = -\varepsilon(v_{i+1} - v_i)/\left(\mathbb{E}[v|v \in N_i] - \mathbb{E}[v|v \in N_{i+1}]\right)$ yields $\tilde{v}_i(\varepsilon)$ and $\tilde{v}_i(\varepsilon)$ for negative ε.

The last step is to rule out that $\mathcal{L}$ has slope 0 in ε (when fixing all variables apart from $\tilde{v}_i(\varepsilon)$ and $\tilde{v}_{i+1}(\varepsilon)$ at their optimal values). To get a contradiction suppose this was the case and note that this is only possible if $\lambda > 0$, see (11) and recall that $Y_B(v_{i+1}) > Y_B(v_i)$ by lemma 3. Then there exists an $\varepsilon' > 0$ such that

$$\tilde{v}_i(\varepsilon') - \left(\tilde{v}_{i+1}(\varepsilon') - \tilde{v}_i(\varepsilon')\right)\frac{\lambda}{1+\lambda}\frac{1-W_i}{\omega_i} = \tilde{v}_{i+1}(\varepsilon') - \left(v_{i+2} - \tilde{v}_{i+1}(\varepsilon')\right)\frac{\lambda}{1+\lambda}\frac{1-W_{i+1}}{\omega_{i+1}}, \tag{12}$$

i.e. the two signals have the same virtual valuation.[12] $\mathcal{L}$ evaluated for ε' is the same as when evaluated at the optimal solution by the assumption that its derivative in ε is zero. As a next step, change $y(v_i, \cdot)$ and $y(v_{i+1}, \cdot)$ by assigining the average trading probability, i.e. $\tilde{y}(\tilde{v}_i, c_j) = \tilde{y}(\tilde{v}_{i+1}, c_j) = y(v_i, c_j)\omega_i/(\omega_i + \omega_{i+1}) + y(v_{i+1}, c_j)\omega_{i+1}/(\omega_i + \omega_{i+1})$ for all $j = 1, \ldots, m$. As both $\tilde{v}_i(\varepsilon')$ and $\tilde{v}_{i+1}(\varepsilon')$ have the same virtual valuation and as $\mathcal{L}$ is linear in y, this does not change the value of $\mathcal{L}$. Finally, note that due to the argument in the proof of lemma 3, merging the two signals $\tilde{v}_i(\varepsilon')$ and $\tilde{v}_{i+1}(\varepsilon')$ into one signal will not affect welfare but relax the budget balance constraint. Hence, such a merging of signals will strictly increase $\mathcal{L}$. But this implies that $(v_i, v_{i+1}, y(v_i, \cdot), y(v_{i+1}, \cdot))$ do not jointly maximize $\mathcal{L}$ in an auxilliary problem in which we fix all other variables at their optimal values. This, however, contradicts the optimality of $(v_i, v_{i+1}, y(v_i, \cdot), y(v_{i+1}, \cdot))$.

The argument for the seller is analogous. □

Proof of lemma 5: Suppose otherwise, i.e. let the optimal information structure put positive probability on types $v_{-i} < v_i < v_{i+1}$ and let v_{i-1} and v_{i+1} be neighboring elements in the support of H_B. Denote the corresponding probabilities in the optimal information structure by ω_{i-1}, ω_i and ω_{i+1}. We will consider the following alternative distributions indexed by ε:

$$\begin{aligned}\tilde{\omega}_{i-1}(\varepsilon) &= \omega_{i-1} - \varepsilon\frac{v_{i+1} - v_i}{v_{i+1} - v_{i-1}} \\ \tilde{\omega}_i(\varepsilon) &= \omega_i + \varepsilon \\ \tilde{\omega}_{i+1}(\varepsilon) &= \omega_{i+1} - \varepsilon\frac{v_i - v_{i-1}}{v_{i+1} - v_{i-1}}.\end{aligned}$$

(All other variables, e.g. cost types probabilities of trade and other valuation types, are fixed at their optimal levels.) Note that the expected valuation is not affected by changes in ε and as v_{i-1} and v_{i+1} are *neighboring* elements of the true valuation support positive as well as negative ε are feasible (if not too large in absolute value).

Now consider the Lagrangian $\mathcal{L}$ of the maximization problem maximizing expected welfare over ε subject to the budget balance constraint (fixing all other variables at their optimal level). From the definition $\tilde{\omega}_{i-1}$, $\tilde{\omega}_i$ and $\tilde{\omega}_{i+1}$, it is clear the $\mathcal{L}$ is linear in ε. As ω_{i-1}, ω_i and ω_{i+1} are by assumption part of the optimal solution, $\mathcal{L}$ has to be maximized by $\varepsilon = 0$. As $\mathcal{L}$ is linear in ε and as ε in an open

[12] To be precise, such an ε' exists as the left hand side (LHS) of (12) is strictly below RHS for $\varepsilon' = 0$, LHS is strictly increasing in ε' while RHS is strictly decreasing in ε' and both LHS and RHS are continuous in ε'. Furthermore, $\tilde{v}_i(\varepsilon') = \tilde{v}_{i+1}(\varepsilon')$ if $\varepsilon' = (\omega_{i+1}\omega_i(v_{i+1} - v_i)/(v_{i+1}\omega_i - v_i\omega_{i+1} + i + 1\omega_{i+1} - v_i\omega_i))$ and RHS<LHS for this ε'. Consequently, the intermediate value theorem implies that an ε' exists at which LHS=RHS.

interval around 0 are feasible, this can only be the case if the derivative of $\mathcal{L}$ with respect to ε is zero everywhere. In the following it is shown that this is not possible.

Suppose the derivative of $\mathcal{L}$ with respect to ε is zero everywhere. For $\varepsilon = 0$, we have $VV(v_{i-1}, 0) < VV(v_i, 0) < VV(v_{i+1}, 0)$ by lemma 3 (where $VV(v_i, \varepsilon)$ denotes the virtual valuation of v_i for a given ε). As ε increases the virtual valuations change as $\tilde{\omega}_{i-1}$ and $\tilde{\omega}_{i+1}$ decrease while ω_i increases. Denote by $\varepsilon' > 0$ the lowest ε such that (at least) one of the following conditions is met

- $VV(v_i, \varepsilon) = VV(v_{i+1}, \varepsilon)$
- $\tilde{\omega}_{i-1}(\varepsilon) = 0$.

For concreteness, let the first condition be met at ε', i.e. $VV(v_i, \varepsilon') = VV(v_{i+1}, \varepsilon')$. Note that the value of $\mathcal{L}$ at $\varepsilon = \varepsilon'$ is the same as at $\varepsilon = 0$ as the derivative of $\mathcal{L}$ with respect to ε is supposed to be zero. As a next step (which will again not change $\mathcal{L}$), change $y(v_i, \cdot)$ and $y(v_{i+1}, \cdot)$ to $\tilde{y}(v_i, c_j) = \tilde{y}(v_{i+1}, c_j) = y(v_i, c_j)\omega_i/(\omega_i + \omega_{i+1}) + y(v_{i+1}, c_j)\omega_{i+1}/(\omega_i + \omega_{i+1})$ for $j = 1, \ldots, m$. This change will not affect $\mathcal{L}$ as $\mathcal{L}$ is linear in $y(v_i, c_j)$ with slope equal to the virtual valuation (plus a term that is constant across buyer signals and therefore unaffected) and both $\tilde{v}_i$ and $\tilde{v}_{i+1}$ had the same virtual valuation. As a last step, note that – following the proof of lemma 3 – merging types v_i and v_{i+1} to $v_i\omega_i/(\omega_i + \omega_{i+1}) + v_{i+1}\omega_{i+1}/(\omega_i + \omega_{i+1})$ with probability $\tilde{\omega}_i(\varepsilon') + \tilde{\omega}_{i+1}(\varepsilon')$ will not affect expected welfare but relax the budget constraint, see the proof of lemma 3. Hence, the value of $\mathcal{L}$ increases due to this change. However, this contradicts that at the optimal solution $\mathcal{L}$ is maximized by the "optimal" values v_{i-1}, v_i, v_{i+1} and ω_{i-1}, ω_i, ω_{i+1} (holding all other variables at their optimal values).

If the other conditions is met at ε', i.e. $\tilde{\omega}_{i-1}(\varepsilon') = 0$, the last step of the proof is similar. If $\tilde{\omega}_{i-1}(\varepsilon') = 0$, eliminating v_{i-1} will strictly increase $\mathcal{L}$ (as v_i's incentive compatibility constraint is strictly relaxed). $\square$

Proof of lemma 6: Consider the hypothetical problem of maximizing expected welfare subject to budget balance being violated by no more than η (through the choice of an information structure and mechanism). Denote the by $W^*(\eta)$ the value of this maximization problem (more formally, the supremum of welfare achievable by information structures and mechanisms that do not violate the ex ante budget balance constraint by more than η). Note that due to the same argument as in the proof of lemma 4 the budget balance constraint binds and therefore W^* is strictly increasing. As both expected welfare and the budget balance condition are continuous in y, W^* is also continuous. Let $\tilde{\eta} < 0$ be such that $W^*(0) - W^*(\tilde{\eta}) < \varepsilon/3$. (Note that a negative η indicates a stricter constraint.)

Define the set of distributions $\mathcal{F}_\kappa$ as the set of distributions with cdfs F_κ such that (i) $\mathbb{E}_{F_\kappa}[v] \leq \mathbb{E}_{H_B}[v] - \kappa$ and (ii) $\int_{-\infty}^{x} F_\kappa(v)\, dv \leq \int_{-\infty}^{x} H_B(v + \kappa)\, dv - \kappa$ for all $x \in (-\infty, \max supp(H_B) - \kappa]$. Similarly, define the set $\mathcal{G}$ as the set of distributions with cdfs G_κ such that (i) $\mathbb{E}_{G_\kappa}[c] \geq \mathbb{E}_{H_S}[c] + \kappa$ and (ii) $\int_{-\infty}^{x} G_\kappa(c)\, dc \leq \int_{-\infty}^{x} H_S(c + \kappa)\, dc - \kappa$ for all $x \in (-\infty, \max supp(H_S) - \kappa]$. Note that $\mathcal{F}_0$ and $\mathcal{G}_0$ are the feasible sets of distributions in the welfare maximization problem of this paper as the set of mean preserving spreads of a distribution equals the set of distributions that have the same mean while also second order stochastically dominating the distribution, see Mas-Colell et al. (1995, ch. 6.D).

Consider now the problem of maximizing welfare subject to budget balance being violated by no more than $\tilde{\eta}$ over the sets $\mathcal{F}_\kappa$ and $\mathcal{G}_\kappa$. Let F and G denote an information structure such that under this information structure and the optimal mechanism (i) budget balance is violated by at most $\tilde{\eta}$, (ii) welfare is above $W^*(\tilde{\eta}) - \varepsilon/3$ and (iii) $F \in \mathcal{F}_{\tilde{\kappa}}$ and $G \in \mathcal{G}_{\tilde{\kappa}}$ for some $\tilde{\kappa} > 0$. Such F, G and $\tilde{\kappa}$ exist by the definition of $\tilde{\eta}$ and as the conditions defining $\mathcal{F}_\kappa$ and $\mathcal{G}_\kappa$ are continuous in κ (while welfare and budget balance constraint are continuous in signals).

Approximate (F, G) by a series of distributions $(F_n, G_n)_{n=1}^{\infty}$ such that (i) the support of F_n and G_n have at most n elements and (ii) $F_n \to F$ almost everywhere and $G_n \to G$ almost everywhere. Then

F_n (G_n) converges to F (G) weakly and by the Helly-Bray theorem welfare and budget balance under (F_n, G_n) converge to the corresponding values under (F, G).[13] Therefore for some sufficiently high n^* welfare under (F_{n^*}, G_{n^*}) is above $W^*(\tilde{\eta}) - 2\varepsilon/3 > W^*(0) - \varepsilon$ and budget balance is violated by at most $\tilde{\eta}$. But this implies – by $\tilde{\eta} < 0$ – that under the finite information structure (F_{n^*}, G_{n^*}) welfare above $W^*(0) - \varepsilon$ is achievable without violating budget balance. Finally, define $F^*_{n^*}$ by "shifting F_{n^*} up" such that $F^*_{n^*}$ has expected value $\mathbb{E}_{H_B}[v]$, i.e. $F^*_{n^*}(x) = F_{n^*}(x - \mathbb{E}_{H_B}[v] + \mathbb{E}_{F_{n^*}}[v])$ and note that the definition of $\mathcal{F}_{\tilde{\kappa}}$ implies $\mathbb{E}_{H_B}[v] - \mathbb{E}_{F_{n^*}}[v] > 0$ (for n^* sufficiently high). Similarly, define $G^*_{n^*}(x) = F_{n^*}(x + \mathbb{E}_{H_S}[c] - \mathbb{E}_{G_{n^*}}[c])$. Note that shifting the distribution of buyer (seller) valuations up (down) by a constant, increases welfare and relaxes the budget balance constraint, see (7). Consequently, welfare under $(F^*_{n^*}, G^*_{n^*})$ is above $W(0) - \varepsilon$. Furthermore, H_B is a mean preserving spread of $F^*_{n^*}$ by the definition of $\mathcal{F}_{\tilde{\kappa}}$ and similarly H_S is a mean preserving spread of $G^*_{n^*}$. Consequently, welfare of at least $W(0) - \varepsilon$ can be achieved by a feasible finite information structure. □

Proof of proposition 2: The proof is by contradiction, i.e. I show that any information structures and mechanism such that $y(v_i, c_j) \in (0, 1)$ are not jointly optimal. To do so consider the problem of maximizing the Lagrangian (8) over y, signals and probabilities. Optimality requires that there is no feasible information structure and mechanism achieving a higher Lagrangian value than the optimal mechanism and information structure. For now, assume that the Lagrange parameter λ is strictly positive. The proof exploits the following intermediate result in a number of ways:

Intermediate Result: In any information structure and mechanism maximizing the Lagrangian, $y(v_i, \cdot) \neq y(v_{i+1}, \cdot)$ for any two buyer signal v_i and v_{i+1}. Similarly, $y(\cdot, c_j) \neq y(\cdot, c_{j+1})$ for any two seller signals.

Proof of the intermediate result: Suppose otherwise, i.e. $y(v_i, \cdot) = y(v_{i+1}, \cdot)$. Now consider an alternative information structure in which the two types v_i and v_{i+1} are "merged", i.e. $\tilde{v} = (\omega_i v_i + \omega_{i+1} v_{i+1})/(\omega_i + \omega_{i+1})$ and $\tilde{\omega} = \omega_i + \omega_{i+1}$ and $y(\tilde{v}, \cdot) = y(v_i, \cdot)$ while all other variables remain as in the supposedly optimal mechanism. Clearly, expected welfare is not affected by the merging of types. However, due to the same steps as in the proof of lemma 3 the budget balance constraint is relaxed by the merging of types. Consequently, the value of the Lagrangian (8) is increased which contradicts the optimality of the original information structure. The proof for the seller is analogous. □

Suppose to the contrary of proposition 2 that $y(v_i, c_j) \in (0, 1)$. Note that this implies that the derivative of $\mathcal{L}$ with respect to $y(v_i, c_j)$ equals zero as $\mathcal{L}$ is linear in $y(v_i, c_j)$. Hence, changing $y(v_i, c_j)$ to either 0 or 1 does not affect the value of the Lagrangian. If such a change results in two adjacent types having the same mechanism y, the intermediate result above implies that optimality is violated as there exists another information structure leading to a strictly higher value of the Lagrangian.

To see that such a change leads to two adjacent types having the same mechanism y, note first that by the monotonicity of the virtual valuation $y(v_i, c_j) < 1$ implies $y(v_i, c_k) = 0$ for all $k > j$ and $y(v_l, c_j) = 0$ for all $l < i$. Furthermore, $y(v_i, c_j) > 0$ implies $y(v_i, c_k) = 1$ for all $k < j$ and $y(v_l, c_j) = 1$ for all $l > i$; see table 4 for an illustration. This implies that if $y(v_{i+1}, c_{j+1}) = 1$, then after changing $y(v_i, c_j)$ to zero $y(\cdot, c_j) = y(\cdot, c_{j+1})$. If, however, $y(v_{i+1}, c_{j+1}) = 0$, then after changing $y(v_i, c_j)$ to 1 $y(v_i, \cdot) = y(v_{i+1}, \cdot)$. If, $y(v_{i+1}, c_{j+1}) \in (0, 1)$, then changing $y(v_{i+1}, c_{j+1})$ to zero and $y(v_i, c_j)$ to 1 will not affect the value of the Lagrangian but then again $y(v_i, \cdot) = y(v_{i+1}, \cdot)$. Finally, observe that if $i = n$ or $j = m$ (and therefore there is not v_{i+1} and c_{j+1}) similar steps can be undertaken with v_{i-1} and c_{j-1} instead of v_{i+1} and c_{j+1}.

Finally, consider $\lambda = 0$. In this case, $y(v_i, c_j) \in (0, 1)$ implies $v_i = c_j$ by (9). Hence, all the steps above (in the $\lambda > 0$ case) will maintain the Lagrangian value and therefore welfare while – through the

[13] We use the same mechanism as under (F, G) here. For completeness, define $y(v, c) = sup_{v' < v, c' > c} y(v', c')$ for all (v, c) not in the support of (F, G) (and let $y(v, c) = 0$ if $y(v', c')$ is not defined for any $v' < v$ and $c' > c$). This ensures the monotonicity of Y_S and Y_B.

	$\cdots$	c_{j-1}	c_j	c_{j+1}	$\cdots$
$\vdots$			$\vdots$	$\vdots$	$\cdots$
v_{i-1}			0	0	$\cdots$
v_i	$\cdots$	1	$y(v_i, c_j)$	0	$\cdots$
v_{i+1}	$\cdots$	1	1		
$\vdots$	$\cdots$	$\vdots$	$\vdots$		

Table 4: Implications of strictly monotone virtual valuation and $y(v_i, c_j) \in (0, 1)$

merging of types – strictly relax the budget balance constraint. The resulting information structure and mechanism would then be optimal while the budget balance constraint would be slack. This is according to lemma 4 only possible if the fully informative information structure attains first best welfare which was ruled out by assumption. □

Proof of lemma 7: The proofs will be by contradiction, i.e. I will show a welfare improvement if the properties do not hold. First, suppose $y(v_l, c_h) > 0$. Note that this implies $y(v_i, c_j) = 1$ for all $(v_i, c_j) \neq (v_l, c_h)$ by monotonicity of the virtual valuation. By (9), $y(v_l, c_h) > 0$ implies $v_l \geq c_h$ (with strict inequality if either $v_h > v_l$ or $c_l < c_h$ have positive probability mass) and therefore welfare would be higher if $y(v_l, c_h) = 1$, i.e. expected welfare would be higher if all buyer types bought from all seller types. As $v_l \geq c_h$ implies $\mathbb{E}[v] \geq \mathbb{E}[c]$ (again with strict inequality if either $v_h > v_l$ or $c_l < c_h$ have positive probability mass), trade with probability 1 is feasible by an information structure that sends signal $\mathbb{E}[v]$ to all buyers and $\mathbb{E}[c]$ to all sellers paired with a fixed price mechanism (where the fixed price is in $[\mathbb{E}[c], \mathbb{E}[v]]$.) Consequently, there is no optimal information structure in which (i) the support of one of the players has two elements and (ii) $y(v_l, c_h) > 0$.

Second, suppose $y(v_h, c_l) < 1$. By monotonicity of the virtual valuation, this implies $y(v_i, c_j) = 0$ for all $(v_i, c_j) \neq (v_h, c_l)$. By (9), $y(v_h, c_l) < 1$ implies $v_h \leq c_l$. This implies that the expected welfare in this mechanism and information structure is at most zero. Hence, it remains to show that there is an alternative information structure and mechanism yielding strictly positive welfare. By the assumption 1, there exists a fixed price t such that the probability that $v \geq t$ as well as the probability that $c \leq t$ is strictly positive. Consider now the information structure that sends a high signal to buyers with valuation weakly above t and a low signal otherwise. Similarly, let the signal for sellers with $c \leq t$ be low and high otherwise. Pair this information structure with a mechanism enforcing trade if and only if the buyers signal is high and the sellers signal is low at price t. Clearly, this mechanism is incentive compatible, individual rational, budget balanced and yields strictly positive welfare. □

Proof of lemma 8: By proposition 2, $y(v_l, c_l)$ and $y(v_h, c_h)$ are in $\{0, 1\}$. To show that trade takes place if and only if expected value is above expected cost note that (9) implies the "only if" part. For "if" consider first the case where either $y(v_l, c_l) = 0$ or $y(v_h, c_h) = 0$ (or both). In these cases, the optimal mechanism is a fixed price mechanism in which the fixed price can be chosen either $t = v_h$ or $t = c_l$ and clearly the result holds. The only remaining case is $y(v_h, c_h) = y(v_l, c_l) = 1$ and it remains to show $c_h > v_l$ in this case. Consider to the contrary $v_l \geq c_h$. But in this case a fixed price contract at price $t = c_h$ and trade with probability 1 would increase welfare while being budget balanced, incentive compatible and individually rational. As $y(v_l, c_h) > 0$ in the optimal information structure was ruled out in lemma 7, the result follows. □

Proof of lemma 9: By lemmas 7 and 8, the only other possibilities are (i) $y(v_l, c_l) = 0 = y(v_l, c_h)$ while $y(v_h, c_l) = 1 = y(v_h, c_h)$, (ii) $y(v_l, c_l) = 1 = y(v_h, c_l)$ while $y(v_l, c_h) = 0 = y(v_h, c_h)$ and (iii) $y(v_l, c_l) = 0 = y(v_h, c_h) = y(v_l, c_h)$ while $y(v_h, c_l) = 1$. In (i) costs are not decision relevant and therefore it is without loss to have only one cost signal. In (ii) valuations are not decision relevant and it is without loss to have only one valuation signal. Hence, these cases are not considered here as the support of the signal structure is not binary for both players. Case (iii) is considered next.

Moving to $y(v_l, c_l) = 0 = y(v_h, c_h) = y(v_l, c_h)$ while $y(v_h, c_l) = 1$, which means that trade occurs only between the high valuation and the low cost type, note that by lemma 8 $v_l \le c_l$ and $v_h \le c_h$. This immediately implies that $c_l > \underline{c}$ and $v_h < \bar{v}$ by assumption 1 and therefore $c_h = \bar{c}$ and $v_l = \underline{v}$ by corollary 2. Next I show $v_h = \bar{c}$. By lemma 8, $y(v_h, c_h) = 0$ implies $v_h \le c_h = \bar{c}$. If $v_h < \bar{c}$, then sending the c_l signal to less $\bar{c}$ types will improve welfare as it reduces the probability of inefficient trade. Hence, $v_h = \bar{c}$. An analogous argument establishes $c_l = \underline{v}$. But in this case there are no gains from trade between a $\underline{v}$ type receiving signal v_h and a seller of signal c_l. Hence, welfare does not change if the buyer receives a fully informative signal (while holding the seller's information structure and y fix). But as $\bar{v} > \bar{c} \ge c_h$ welfare can be strictly increased from there by changing y to $y(v_h, c_h) = 1$.

Next consider $y(v_l, c_l) = 1 = y(v_h, c_h)$, which implies that trade happens unless the cost signal is high and the valuation signal is low. I will hold the mechanism, i.e. y, fixed for the remainder of the proof and first focus on the buyer showing that $v_h = \bar{v}$ in the optimal information structure. By way of contradiction suppose $v_h < \bar{v}$ and note that by corollary 2 this implies $v_l = \underline{v}$. As $v_h < \bar{v}$, some buyers with true valuation $\underline{v}$ receive the signal v_h. Consider now moving ε of these buyers to signal v_l. Put differently, the following information structures are feasible for small $\varepsilon > 0$:

$$\tilde{v}_l(\varepsilon) = \underline{v} \qquad \tilde{v}_h(\varepsilon) = \frac{\omega_h - \varepsilon - \bar{\omega}}{\omega_h - \varepsilon}\underline{v} + \frac{\bar{\omega}}{\omega_h - \varepsilon}\bar{v} \qquad \tilde{\omega}_h(\varepsilon) = \omega_h - \varepsilon \qquad \tilde{\omega}_l(\varepsilon) = 1 - \omega_h + \varepsilon$$

where $\bar{\omega}$ is the share of $\bar{v}$ in the true buyer type distribution. Note that the original information structure is obtained for $\varepsilon = 0$. The budget balance condition in the binary case (with y as fixed above) can be written as

$$\gamma_l \tilde{v}_l(\varepsilon) + (1 - \gamma_l)\tilde{\omega}_h(\varepsilon)\tilde{v}_h(\varepsilon) - \tilde{\omega}_h(\varepsilon)c_h - \gamma_l(1 - \tilde{\omega}_h(\varepsilon))c_l \ge 0.$$

The derivative of the left hand side of this budget balance condition with respect to ε is $c_h - \underline{v} + \gamma_l(\underline{v} - c_l)$ which is positive as $c_h \ge \underline{v}$ and $\underline{v} \ge c_l$ by lemmas 7 and 8. Clearly expected welfare is strictly increasing in ε as well as the moved types with valuation $\underline{v}$ no longer trade inefficiently with high cost sellers. This implies that expected welfare is strictly higher for $\varepsilon > 0$ while budget balance is not violated and thereby optimality of the original information structure is contradicted. Hence, $v_h = \bar{v}$ has to hold in the optimal information structure.

The proof for $c_l = \underline{c}$ in the optimal information structure is analogous. □

Proof of proposition 3: See appendix D below.

D. Derivations binary type distribution

By lemmas 7 and 9, $y(v_h, c_h) = y(v_l, c_l) = y(v_h, c_l) = 1$ while $y(v_l, c_h) = 0$ and $v_h = \bar{v}$ while $c_l = \underline{c}$. Let $\bar{\omega}$ ($\underline{\gamma}$) be the share of high (low) types in H_B (H_S). Then the optimization problem can be formulated in terms of the variables $\omega_h \in [0, \bar{\omega}]$ and $\gamma_l[0, \underline{\gamma}]$ and

$$\begin{aligned} v_l &= \frac{\bar{\omega} - \omega_h}{1 - \omega_h}\bar{v} + \frac{1 - \bar{\omega}}{1 - \omega_h}\underline{v} \\ c_h &= \frac{\underline{\gamma} - \gamma_l}{1 - \gamma_l}\underline{c} + \frac{1 - \underline{\gamma}}{1 - \gamma_l}\bar{c}. \end{aligned}$$

The budget balance constraint can be written as

$$BB(\omega_h, \gamma_l) = \gamma_l \frac{\bar{\omega} - \omega_h}{1 - \omega_h}\bar{v} + \gamma_l \frac{1 - \bar{\omega}}{1 - \omega_h}\underline{v} + (1 - \gamma_l)\omega_h \bar{v} - \omega_h \frac{\underline{\gamma} - \gamma_l}{1 - \gamma_l}\underline{c} - \omega_h \frac{1 - \underline{\gamma}}{1 - \gamma_l}\bar{c} - \gamma_l(1 - \omega_h)\underline{c} \ge 0.$$

The objective, expected welfare, equals

$$W(\omega_h,\gamma_l)=(\omega_h\underline{\gamma}+(\bar{\omega}-\omega_h)\gamma_l)(\bar{v}-\underline{c})+\gamma_l(1-\bar{\omega})(\underline{v}-\underline{c})+\omega_h(1-\underline{\gamma})(\bar{v}-\bar{c}).$$

As $W(\omega_h,\gamma_l)$ is strictly increasing in both variables, the budget balance constraint holds with equality if and only if $BB(\bar{\omega},\underline{\gamma})<0$: If BB held with inequality, increasing either γ_l or ω_h by a sufficiently small amount would increase welfare without violating BB.

Note at this point that it is possible to normalize the problem as described in the main text: the maximizing ω_h and γ_l in the original problem equal the maximizing choices in the normalized problem in which $\bar{v}^{normal}=1$, $\underline{v}^{normal}=\underline{v}/\bar{v}$, $\underline{c}^{normal}=\underline{c}/\bar{v}$ and $\bar{c}^{normal}=\bar{c}/\bar{v}$. First/second best welfare in the original problem equals first/second best welfare in the normalized problem times $\bar{v}$. This is true as W, W^{fb}, and BB are linear in the types $\bar{v}$, $\underline{v}$, $\bar{c}$ and $\underline{c}$.

Solving the budget balance condition for ω_h yields[14]

$$\omega_h^{BB}(\gamma_l)=\frac{1}{2}\left(1+\frac{\gamma_l(\bar{v}-\underline{c})}{\frac{1-\underline{\gamma}}{1-\gamma_l}(\bar{c}-\underline{c})-(1-\gamma_l)(\bar{v}-\underline{c})}\right)$$
$$-\sqrt{\frac{1}{4}\left(1+\frac{\gamma_l(\bar{v}-\underline{c})}{\frac{1-\underline{\gamma}}{1-\gamma_l}(\bar{c}-\underline{c})-(1-\gamma_l)(\bar{v}-\underline{c})}\right)^2-\frac{\gamma_l\bar{\omega}(\bar{v}-\underline{v})+\gamma_l(\underline{v}-\underline{c})}{\frac{1-\underline{\gamma}}{1-\gamma_l}(\bar{c}-\underline{c})-(1-\gamma_l)(\bar{v}-\underline{c})}}$$

while solving the budget balance condition for γ_l yields

$$\gamma_l^{BB}(\omega_h)=\frac{1}{2}\left(1+\frac{\omega_h(\bar{v}-\underline{c})}{\frac{1-\bar{\omega}}{1-\omega_h}(\bar{v}-\underline{v})-(1-\omega_h)(\bar{v}-\underline{c})}\right)$$
$$-\sqrt{\frac{1}{4}\left(1+\frac{\omega_h(\bar{v}-\underline{c})}{\frac{1-\bar{\omega}}{1-\omega_h}(\bar{v}-\underline{v})-(1-\omega_h)(\bar{v}-\underline{c})}\right)^2-\frac{\omega_h(\bar{v}-\bar{c})+\omega_h\underline{\gamma}(\bar{c}-\underline{c})}{\frac{1-\bar{\omega}}{1-\omega_h}(\bar{v}-\underline{v})-(1-\omega_h)(\bar{v}-\underline{c})}}.$$

$\omega_h^{BB}(\gamma_l)$ can be plugged into W in order to get a one-dimensional optimization problem over $\gamma_l\in[\gamma_l^{BB}(\bar{\omega}),\underline{\gamma}]$. I numerically verified that the resulting objective function is convex in γ_l (under the assumption that $BB(\bar{\omega},\underline{\gamma})<0$).[15] Consequently the solution is either

- $\gamma_l=\gamma_l^{BB}(\bar{\omega})$ and therefore $\omega_h=\bar{\omega}$ or
- $\gamma_l=\underline{\gamma}$ and therefore $\omega_h=\omega_h^{BB}(\underline{\gamma})$.

Put differently, one player receives a perfectly informative signal and the other player a noisy signal. For concreteness, the relevant values $\gamma_l^{BB}(\bar{\omega})$ and $\omega_h^{BB}(\underline{\gamma})$ are given explicitly:

$$\begin{aligned}
\gamma_l^{BB}(\bar{\omega}) &= \frac{1}{2}\left(1+\frac{\bar{\omega}(\bar{v}-\underline{c})}{\bar{\omega}\bar{v}-\underline{v}+(1-\bar{\omega})\underline{c}}\right)-\sqrt{\frac{1}{4}\left(1+\frac{\bar{\omega}(\bar{v}-\underline{c})}{\bar{\omega}\bar{v}-\underline{v}+(1-\bar{\omega})\underline{c}}\right)^2-\frac{\bar{\omega}(\bar{v}-\bar{c})+\bar{\omega}\underline{\gamma}(\bar{c}-\underline{c})}{\bar{\omega}\bar{v}-\underline{v}+(1-\bar{\omega})\underline{c}}}\\
\omega_h^{BB}(\underline{\gamma}) &= \frac{1}{2}\left(1+\frac{\underline{\gamma}(\bar{v}-\underline{c})}{\bar{c}-\underline{\gamma}\underline{c}-(1-\underline{\gamma})\bar{v}}\right)-\sqrt{\frac{1}{4}\left(1+\frac{\underline{\gamma}(\bar{v}-\underline{c})}{\bar{c}-\underline{\gamma}\underline{c}-(1-\underline{\gamma})\bar{v}}\right)^2-\frac{\underline{\gamma}\bar{\omega}(\bar{v}-\underline{v})+\underline{\gamma}(\underline{v}-\underline{c})}{\bar{c}-\underline{\gamma}\underline{c}-(1-\underline{\gamma})\bar{v}}}.
\end{aligned}$$

To determine which of the two solutions yields a higher welfare it is simplest to compare for both

[14] The second solution of the quadratic equation is above 1 – as $\frac{\gamma_l(\bar{v}-\underline{c})}{\frac{1-\underline{\gamma}}{1-\gamma_l}(\bar{c}-\underline{c})-(1-\gamma_l)(\bar{v}-\underline{c})}>1$ by $\gamma_l\leq\underline{\gamma}$ – and therefore not relevant. Note that there always exists a solution in $(0,1)$ as the budget balance constraint is slack if $\omega_h=0$.

[15] The code is available on the website of the author (`https://schottmueller.github.io/`).

the difference to first best welfare. As $W(\bar{\omega}, \gamma_l)$ is linear in γ_l this difference can be expressed as

$$W(\bar{\omega}, \underline{\gamma}) - W(\bar{\omega}, \gamma_l^{BB}(\bar{\omega})) = (1-\bar{\omega})(\underline{v} - \underline{c})$$
$$\left(\underline{\gamma} - \frac{1}{2}\left(1 + \frac{\bar{\omega}(\bar{v} - \underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{\bar{\omega}(\bar{v} - \underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}\right)^2 - \frac{\bar{\omega}(\bar{v} - \bar{c}) + \bar{\omega}\underline{\gamma}(\bar{c} - \underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}}\right)$$

$$W(\bar{\omega}, \underline{\gamma}) - W(\omega_h^{BB}(\underline{\gamma}), \underline{\gamma})) = (1-\underline{\gamma})(\bar{v} - \bar{c})$$
$$\left(\bar{\omega} - \frac{1}{2}\left(1 + \frac{\underline{\gamma}(\bar{v} - \underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{\underline{\gamma}(\bar{v} - \underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}\right)^2 - \frac{\underline{\gamma}\bar{\omega}(\bar{v} - \underline{v}) + \underline{\gamma}(\underline{v} - \underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}}\right).$$

Consequently, $\gamma_l = \gamma_l^{BB}(\bar{\omega})$ and therefore $\omega_h = \bar{\omega}$ in the optimal mechanism if and only if

$$\frac{(1-\underline{\gamma})(\bar{v} - \bar{c})}{(1-\bar{\omega})(\underline{v} - \underline{c})}\left(\bar{\omega} - \frac{1}{2}\left(1 + \frac{\underline{\gamma}(\bar{v} - \underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{\underline{\gamma}(\bar{v} - \underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}\right)^2 - \frac{\underline{\gamma}\bar{\omega}(\bar{v} - \underline{v}) + \underline{\gamma}(\underline{v} - \underline{c})}{\bar{c} - \underline{\gamma}\underline{c} - (1-\underline{\gamma})\bar{v}}}\right)$$
$$\geq \left(\underline{\gamma} - \frac{1}{2}\left(1 + \frac{\bar{\omega}(\bar{v} - \underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}\right) + \sqrt{\frac{1}{4}\left(1 + \frac{\bar{\omega}(\bar{v} - \underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}\right)^2 - \frac{\bar{\omega}(\bar{v} - \bar{c}) + \bar{\omega}\underline{\gamma}(\bar{c} - \underline{c})}{\bar{\omega}\bar{v} - \underline{v} + (1-\bar{\omega})\underline{c}}}\right)$$

and $\gamma_l = \underline{\gamma}$ and therefore $\omega_h = \omega_h^{BB}(\underline{\gamma})$ otherwise.

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