Supplementary material to
"Procurement with specialized firms"

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In this supplementary material, we relax some of the assumptions made in the paper and illustrate how to derive the optimal mechanism in these cases. First, we consider violations of the monotone hazard rate property (or violations of assumption 2 regarding the sign $c_{q\theta\theta}$) under which the first order approach is no longer valid. In section 2, we consider violations of assumption 3. That is, we allow first best welfare to be (locally) maximal at interior types.

1. Second order and global incentive compatibility

In the main paper, we made assumptions on third derivatives of the cost function and the distribution of θ. This allowed us to use a first order approach and ignore global IC constraints. In this section, we introduce the global IC constraint. Then we relax the assumptions of the paper such that the solution of the relaxed problem is not necessarily incentive compatible. We show an ironing procedure that can deal with violations of second order incentive compatibility. Finally, we present a family of cost functions for which the first and second order condition for IC (which are both local) imply global IC.

A menu (q, x, t) is IC in a global sense if and only if

$$\Phi(\hat{\theta}, \theta) \equiv \pi(\theta, \theta) - \pi(\hat{\theta}, \theta) \geq 0 \tag{S1}$$

for all $\theta, \hat{\theta} \in [\underline{\theta}, \bar{\theta}]$; where $\pi(\hat{\theta}, \theta)$ is defined in equation (3) of the main paper.

Equation (4) in the main paper gives the first order condition for IC. This is not sufficient for (even) a local maximum. The relevant second order condition can be written as follows.

Lemma S1. *Second order incentive compatibility requires*

$$X_\theta(\theta)c_\theta(q(\theta),\theta) + X(\theta)c_{q\theta}(q(\theta),\theta)q_\theta(\theta) \leq 0. \tag{SOC}$$

Proof. Define the function

$$\Phi(\hat{\theta},\theta) = \pi(\theta,\theta) - \pi(\hat{\theta},\theta) \geq 0$$

By IC this function is always positive and equal to zero if $\hat{\theta} = \theta$. In other words, the function Φ reaches a minimum at $\hat{\theta} = \theta$. Thus truth-telling implies both

$$\left.\frac{\partial \Phi(\hat{\theta},\theta)}{\partial \hat{\theta}}\right|_{\hat{\theta}=\theta} = 0 \tag{S2}$$

and

$$\left.\frac{\partial^2 \Phi(\hat{\theta},\theta)}{\partial \hat{\theta}^2}\right|_{\hat{\theta}=\theta} \geq 0 \tag{S3}$$

Since equation (S2) has to hold for all $\hat{\theta} = \theta$, differentiating gives

$$\left.\frac{\partial^2 \Phi(\hat{\theta},\theta)}{\partial \hat{\theta}^2}\right|_{\hat{\theta}=\theta} + \left.\frac{\partial^2 \Phi(\hat{\theta},\theta)}{\partial \hat{\theta}\partial\theta}\right|_{\hat{\theta}=\theta} = 0.$$

Then equation (S3) implies that

$$\left.\frac{\partial^2 \Phi(\hat{\theta},\theta)}{\partial \hat{\theta}\partial\theta}\right|_{\hat{\theta}=\theta} \leq 0.$$

It follows from the definition of Φ that

$$\left.\frac{\partial^2 \Phi(\hat{\theta},\theta)}{\partial \hat{\theta}\partial\theta}\right|_{\hat{\theta}=\theta} = X_\theta(\theta)c_\theta(q(\theta),\theta) + X(\theta)c_{q\theta}(q(\theta),\theta)q_\theta(\theta) \leq 0$$

which is the inequality in the lemma. *Q.E.D.*

As shown in textbooks like Laffont and Tirole (1993), first and second order conditions for IC imply global IC (as in equation (S1)) if $c_\theta < 0$ for all $q \in \mathbb{R}_+$. Because we assume that firms are specialized (assumption 2), local IC does not automatically imply global IC. Hence, we still need to verify global IC even if (4) and (SOC) are satisfied.

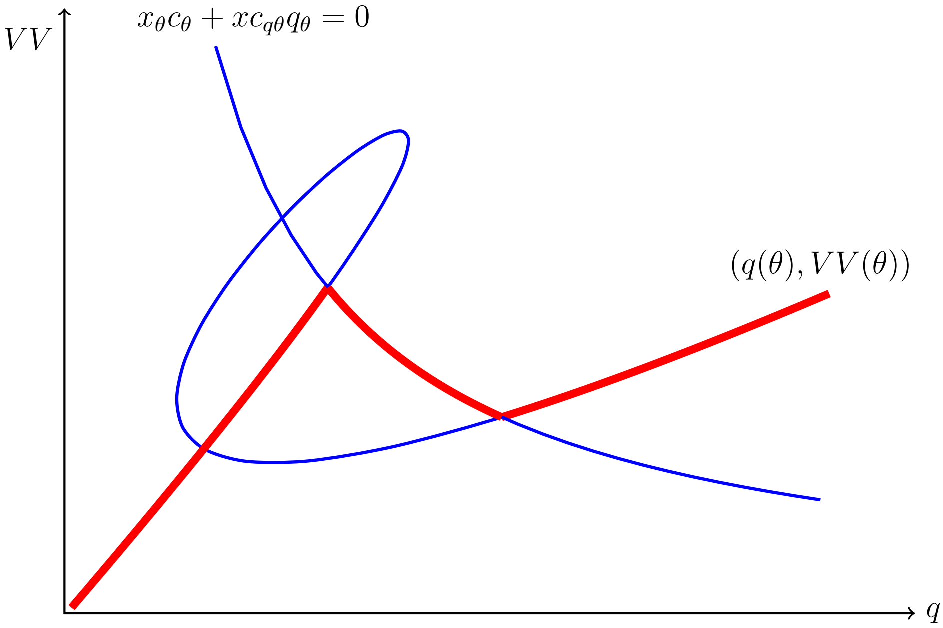

Figure S1: Solution for quality $q(\theta)$ and virtual valuation $VV(\theta)$ for the case where (second order) condition (SOC) is violated.

How should the solution to the relaxed problem be adapted if it is not globally IC because our assumptions are not satisfied? For concreteness, we focus here on the WM case and assume that the problems arise because of a violation of the MHR assumption. The cases where third derivatives cause problems with (SOC) are dealt with analogously. In the WM case, the change in q for $\theta > \theta_b$ is given by

$$q_\theta(\theta) = \frac{c_{q\theta}(q(\theta), \theta) - c_{q\theta\theta}(q(\theta), \theta)\frac{1-F(\theta)}{f(\theta)} - c_{q\theta}(q(\theta), \theta)\frac{d\left(\frac{1-F(\theta)}{f(\theta)}\right)}{d\theta}}{S_{qq}(q(\theta)) - c_{qq}(q(\theta), \theta) + c_{qq\theta}(q(\theta), \theta)\frac{1-F(\theta)}{f(\theta)}}. \tag{S4}$$

Now we consider the case where $d((1 - F(\theta))/f(\theta))/d\theta > 0$ for $\theta > \theta_b$ in such a way that $q_\theta < 0$ and such that $q_\theta < 0$ causes a violation of (SOC). We first sketch how this is dealt with in general. Then we work out an example.

In the one-dimensional case, a violation of (SOC) is dealt with by bunching types on one quality level q; see Guesnerie and Laffont (1984). In the two-dimensional case (q and X), it is not necessarily true that a violation of (SOC) leads to bunching of types θ on the same quality q and probability of winning X. Below we do not work with X but with the virtual valuation VV as there is a one-to-one relation between the two (i.e. higher VV implies higher X and the other way around).

Now, we explicitly add constraint (SOC) to the optimal control problem stated in lemma 2 of the paper: As q_θ is part of the second order condition, the optimal control problem has $y = q_\theta$ as control variable and q and π as state variables. The Lagrangian of this problem is

$$\begin{aligned}\mathcal{L} &= f(\theta)[X(\theta)(S(q(\theta)) - c(q(\theta),\theta)) - \pi(\theta)] \\ &+ \lambda(\theta)(\pi_\theta(\theta) + X(\theta)c_\theta(q(\theta),\theta)) \\ &+ \xi(\theta)(q_\theta(\theta) - y(\theta)) \\ &- \mu(\theta)(X_\theta(\theta)c_\theta(q(\theta),\theta) + X(\theta)c_{q\theta}(q(\theta),\theta)y(\theta)) \\ &+ \eta(\theta)\pi(\theta)d\theta.\end{aligned} \tag{S5}$$

In the optimum, the following first order conditions have to be satisfied:

$$\frac{\partial \mathcal{L}}{\partial y} = -\xi(\theta) - \mu(\theta)X(\theta)c_{q\theta}(q(\theta),\theta) = 0 \tag{S6}$$

$$\begin{aligned}\xi_\theta(\theta) = \frac{\partial \mathcal{L}}{\partial q} &= X(\theta)\left[f(\theta)(S_q(q(\theta) - c(q(\theta),\theta)) + \lambda(\theta)c_{q\theta}(q(\theta),\theta) - \mu(\theta)c_{qq\theta}(q(\theta),\theta)y(\theta)\right] \\ &\quad -\mu(\theta)c_{q\theta}(q(\theta),\theta)X_\theta(\theta)\end{aligned} \tag{S7}$$

$$\lambda_\theta(\theta) = \frac{\partial \mathcal{L}}{\partial \pi} = -f(\theta) + \eta(\theta) \tag{S8}$$

Differentiating (S6) gives

$$\begin{aligned}\xi_\theta(\theta) &= -\mu_\theta(\theta)X(\theta)c_{q\theta}(q(\theta),\theta) - \mu(\theta)X_\theta(\theta)c_{q\theta}(q(\theta),\theta) - \mu(\theta)X(\theta)c_{q\theta\theta}(q(\theta),\theta) \\ &\quad -\mu(\theta)X(\theta)c_{qq\theta}(q(\theta),\theta)q_\theta(\theta).\end{aligned}$$

Plugging this expression into (S7) gives (after cancelling terms and dividing by X) the following first order condition for q:

$$f(\theta)(S_q(q(\theta)) - c_q(q(\theta),\theta)) + \lambda(\theta)c_{q\theta}(q(\theta),\theta) + \mu(\theta)c_{q\theta\theta}(q(\theta),\theta) = -\mu_\theta(\theta)c_{q\theta}(q(\theta),\theta). \tag{S9}$$

Consider figure S1 to illustrate the procedure. This figure shows equation (SOC) (where it holds with equality) in (q, VV) space and the solution $(q(\theta), VV(\theta))$ that follows from the planner's optimization problem while ignoring the second order condition; i.e. assuming $\mu_\theta(\theta) = 0$ for all θ. The former curve is downward sloping in the WM case since

$$\frac{dX}{dq} = \frac{X_\theta(\theta)}{q_\theta(\theta)} = -X(\theta)\frac{c_{q\theta}(q(\theta),\theta)}{c_\theta(q(\theta),\theta)} < 0.$$

In the simple case (that we also use in the example below) where $c_{\theta\theta} = 0$, this curve boils down to

$$X(\theta)c_\theta(q(\theta), \theta) = -K < 0 \tag{S10}$$

for some constant $K > 0$, as differentiating equation (S10) with respect to θ indeed gives the constraint $X_\theta c_\theta + X c_{q\theta} q_\theta = 0$.

The solution of the relaxed program $(q(\theta), VV(\theta))$ (ignoring the second order constraint!), starts at $\underline{\theta}$ in the bottom left corner and moves first over the thick (red) part of this curve, then follows the thin (blue) part, curving back (i.e. both q and x fall with θ) then both q and x increase again with θ and we end on the thick (red) part of the curve. The part of the curve where $q_\theta, X_\theta < 0$ violates (SOC).

Hence, we need to find θ_a where (SOC) starts to bind. Then from θ_a onwards, we follow the binding constraint till we arrive at θ_b, from which point onwards we follow the solution $(q(\theta), VV(\theta))$ again. As shown in figure S1, the choice of θ_a determines both the trajectory $(\tilde{q}(\theta), \tilde{VV}(\theta))$ satisfying equation (SOC) and the end point of this trajectory θ_b. Since $\mu(\theta) = 0$ both for $\theta < \theta_a$ and for $\theta > \theta_b$, it must be the case that $\int_{\theta_a}^{\theta_b} \mu_\theta(\theta) d\theta = 0$. To illustrate, for the case where $c_{q\theta\theta} = 0$,[1] this can be written as (using equation (S9))[2]

$$\int_{\theta_a}^{\theta_b} \frac{f(\theta)(S_q(q(\theta)) - c_q(\tilde{q}(\theta), \theta)) + (1 - F(\theta))c_{q\theta}(\tilde{q}(\theta), \theta)}{c_{q\theta}(\tilde{q}(\theta), \theta)} d\theta = 0. \tag{S11}$$

We now illustrate this approach with an example.

Example 1. *To violate the monotone hazard rate assumption we use the density* $f(\theta) = (\theta - a)^2 + 1/50$ *with support* $[0, a + 1/4]$ *where* a *has to be approximately* 1.42 *to satisfy the requirements of a probability distribution. The hazard rate of this distribution is depicted in figure S2.*

[1] If $c_{q\theta\theta} \neq 0$, the differential equation (S9) has to be solved for $\mu(\theta)$. Although a bit tedious, this is do-able since the differential equation is linear and first order in $\mu(\theta)$.

[2] We immediately plug in $\lambda(\theta) = 1 - F(\theta)$. The reason is that (SOC) cannot bind at types that have zero profits in the relaxed solution: For these types, $q(\theta) = k(\theta)$ in the relaxed solution and therefore $c_\theta(q(\theta), \theta) = 0$ and $q_\theta(\theta) = k_\theta(\theta) > 0$. This implies that (SOC) is strictly slack in the relaxed solution for types with zero profits. Hence, θ_a and θ_b cannot be in the zero profit interval. Consequently, both θ_a and θ_b (and therefore also types in between) are in the part of the relaxed solution where $\lambda(\theta) = 1 - F(\theta)$.

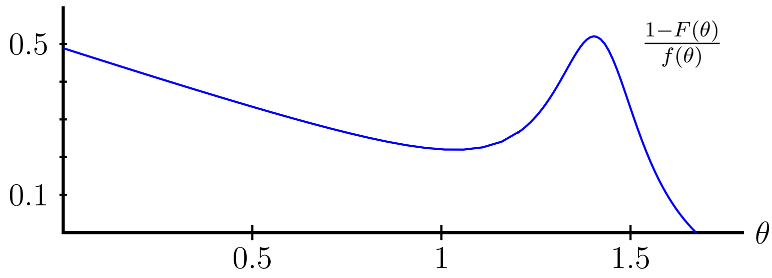

Figure S2: Inverse hazard rate with $f(\theta) = (\theta - a)^2 + 1/50$

Assume that there are two firms, that $S(q) = q$ *and that* $c(q, \theta) = \frac{1}{2}q^2 - q\theta + \theta$. *Then* $c_\theta(q, \theta) = 1 - q$ *which changes sign at* $q = 1$. *As* $c_{\theta\theta} = 0$, *the binding second order condition takes the form of (S10):*

$$X = \frac{K}{q-1}$$

for some $K > 0$. *Note that this equation does not depend on* θ. *Hence, in this case, "following the constraint" takes the form of bunching* $\theta \in [\theta_a, \theta_b]$ *on some point*

$$(\tilde{q}, \tilde{VV}) \tag{S12}$$

where $\tilde{VV}$ *corresponds to the probability* $\tilde{X} = \frac{K}{\tilde{q}-1}$. *Choosing* θ_a, *fixes* $\tilde{q} = q(\theta_a)$ *and* θ_b *since* $q(\theta_b) = \tilde{q}$. *Writing the dependency of* $\tilde{q}, \theta_b$ *on* θ_a *explicitly,* θ_a *solves equation (S11):*

$$\int_{\theta_a}^{\theta_b(\theta_a)} f(\theta)(1 - (\tilde{q}(\theta_a) - \theta)) - (1 - F(\theta))d\theta = 0. \tag{S13}$$

Since equation (SOC) will already start to bind for θ_a *where* $q_\theta(\theta_a) > 0$, *it is routine to verify that this equation is downward sloping in* θ_a. *The unique solution in this example is* $\theta_a \approx 1.1685$ *which gives a corresponding* $\theta_b = 1.428$ *and* $\tilde{q} = 1.923$.

While the ironing procedure described above takes care of the local second order condition (SOC), this does not necessarily imply global incentive compatibility. Global constraints are mathematically intractable in general frameworks; see Araujo and Moreira (2010) and Schottmüller (2011) for special examples of how to handle global constraints. However, the following proposition establishes that global constraints do not bind for a family of cost functions. This family includes the functions we used in the example and the most commonly used linear-quadratic cost functions.

Proposition S1. *If $c_{\theta\theta} = 0$ and the local second order condition (SOC) is satisfied, the solution is globally incentive compatible.*

Proof. As shown in the proof of lemma 3 in the main paper, incentive compatibility between θ and $\hat{\theta}$ boils down to the inequality

$$\int_{\theta}^{\hat{\theta}} \int_{t}^{\hat{\theta}} X_{\theta}(s)c_{\theta}(q(s), t) + X(s)c_{\theta q}(q(s), t)q_{\theta}(s)\; ds\, dt \leq 0.$$

Now note that $c_{\theta\theta} = 0$ implies

$$X_{\theta}(s)c_{\theta}(q(s), t) + X(s)c_{\theta q}(q(s), t)q_{\theta}(s) = X_{\theta}(s)c_{\theta}(q(s), s) + X(s)c_{\theta q}(q(s), s)q_{\theta}(s).$$

But then global incentive compatibility has to be satisfied as $X_s(s)c_{\theta}(q(s), s)+X(s)c_{\theta q}(q(s), s)$ $q_s(s) \leq 0$ by the local second order condition. *Q.E.D.*

2. Best interior type: Solutions if assumption 3 is violated

This material illustrates the optimal mechanism if assumption 3 in the paper does not hold. Without assumption 3 the first best quality q^{fb} can intersect k at several types and first best welfare is not necessarily quasiconvex in type. The only qualitative differences to the optimal mechanism in the paper are that (i) there can be more than one interval of types with zero expected rents and (ii) there can be virtual valuation maxima at interior types.

It is important to note that the proof of lemma 2 does not depend on assumption 3. Hence, lemma 2 still applies. We will focus again on solving the relaxed program and check global incentive compatibility ex post.

We will now solve an example that illustrates how a failure of assumption 3 changes the optimal mechanism. In the example, first best welfare is W-shaped and we can get two zero profit intervals. After solving the relaxed program, we show a monotonicity result and check global incentive compatibility.

Example Let $S(q) = q$, $n = 2$ and $c(q, \theta) = q^2/2 + (1 - \theta)q + \frac{1}{\alpha}\log(e^{\alpha/2} + e^{a\theta})$.

Assume types are uniformly distributed on $[0, 1]$. This implies (see figure S3)

$$\begin{aligned} q^{fb}(\theta) &= \theta \\ k(\theta) &= \frac{1}{1 + e^{-\alpha(\theta - 1/2)}} \\ q^{l}(\theta) &= 2\theta \\ q^{h}(\theta) &= 2\theta - 1. \end{aligned}$$

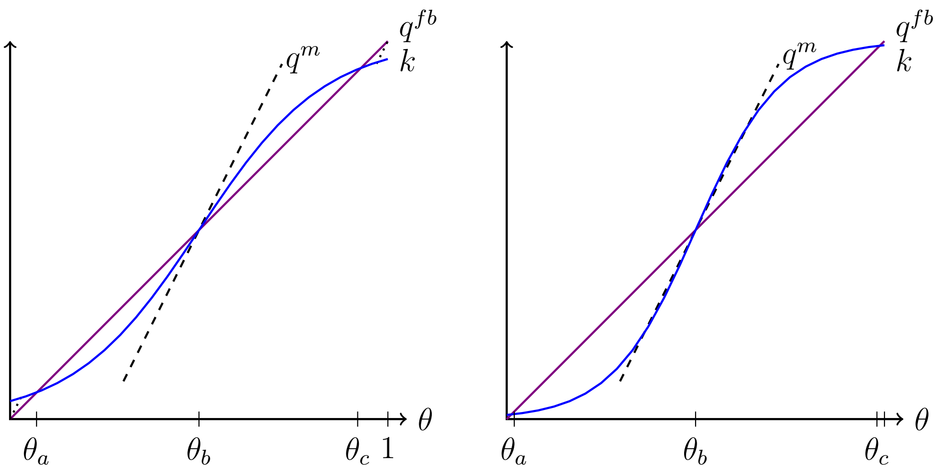

Figure S3: Example with $\alpha = 6$ and $\alpha = 9$

The new feature in this example is that there is an interior type (θ_b in figure S3) which locally maximizes first best welfare. Also there are two interior types locally minimizing first best welfare (θ_a and θ_c).

In line with the solution in the paper, one will expect that the solution for high types (around 1) is q^h and the solution for low types (around 0) is q^l. At the intersection of q^l (q^h) with k a zero profit interval starts (ends). As in the paper, we denote these intersection types θ_1 and θ_2. Because first best welfare is not quasiconvex, it seems, however, possible that some types in (θ_1, θ_2) have positive profits. If this is true, there will be a type θ' in (θ_1, θ_2) such that profits attain a local maximum at this type. Hence, $\pi_\theta = -X(\theta')c_\theta(q(\theta'), \theta') = 0$ at this local profit maximizer. We conclude that $q(\theta') = k(\theta')$. As θ' maximizes profits and $\pi_\theta = -Xc_\theta$ by first order incentive compatibility, types slightly below θ' must have $q(\theta) > k(\theta)$ and types slightly above θ' will have

$q(\theta) < k(\theta)$. The first order condition

$$S_q(q) - c_q(q(\theta), \theta) + \lambda(\theta) c_{q\theta}(q(\theta), \theta) = 0 \tag{S14}$$

can be solved for $\lambda(\theta')$ as we know $q(\theta') = k(\theta')$. In our example, $\lambda(\theta') = k(\theta') + \theta'$. Also the first order condition $\lambda_\theta(\theta) = -f(\theta) + \eta(\theta)$ has to be satisfied. Given that types around θ' have positive profits, we have $\eta(\theta) = 0$ for those types. This implies $\lambda(\theta) = k(\theta') + 2\theta' - \theta$ for θ close to θ'. Plugging this into (S14), we obtain q for the types within (θ_1, θ_2) that have positive profits:

$$q^m(\theta) = 2\theta - k(\theta') - 2\theta'.$$

The interval of types with zero profits within (θ_1, θ_2) has to go from the first to the third intersection type of q^m with k (the second intersection is θ'). Call these two intersection types θ_1' and θ_2'. Hence, the two zero profit intervals are $[\theta_1, \theta_1']$ and $[\theta_2, \theta_2']$. θ' is then defined by the continuity of profits, i.e. $\int_{\theta_1'}^{\theta_2'} -x(\theta) c_\theta(q^m(\theta), \theta)\, d\theta = 0$.

Following our conjecture the first zero profit interval would start at θ_1 and end where q^m intersects with k for the first time. The second zero profit interval would start at the third intersection of q^m and k and end at θ_2. However, it is clear from figure S3 that the number of intersection points between q^m and k depends on α because α influences the slope of k. It turns out that for $\alpha \leq 8$ q^m is steeper than k at every type. Hence, k and q^m can only have one intersection point. Only for $\alpha > 8$ there are three intersection points and the solution structure with two zero profit intervals as described above is feasible.

Let's first analyze the case where $\alpha = 9$. In our example, everything is symmetric around 1/2. Hence, it is not surprising that $\theta' = 1/2$. It then follows that q^m intersects k at the types $\theta_1' = 0.362$, $\theta' = 0.5$ and $\theta_2' = 0.638$. The intersection of q^l and k is $\theta_1 = 0.006$ and the intersection of q^h and k is $\theta_2 = 0.994$. Hence, we obtain the solution candidate

$$q(\theta) = \begin{cases} 2\theta & \text{for } \theta \in [0, \theta_1] \\ \frac{1}{1+e^{-9(\theta-1/2)}} & \text{for } \theta \in (\theta_1, \theta_1'] \cup (\theta_2', \theta_2] \\ 2\theta - \frac{1}{2} & \text{for } \theta \in [\theta_1', \theta_2') \\ 2\theta - 1 & \text{for } \theta \in [\theta_2, 1] \end{cases}$$

together with the costate[3]

$$\lambda(\theta) = \begin{cases} -\theta & \text{for } \theta \in [0, \theta_1] \\ \theta - \frac{1}{1+e^{-9(\theta-1/2)}} & \text{for } \theta \in (\theta_1, \theta_1'] \cup (\theta_2', \theta_2] \\ \frac{1}{2} - \theta & \text{for } \theta \in [\theta_1', \theta_2') \\ 1 - \theta & \text{for } \theta \in [\theta_2, 1] \end{cases}$$

and the Lagrange multiplier of the participation constraint[4]

$$\eta(\theta) = \begin{cases} 2 - \frac{9e^{-9(\theta-1/2)}}{(1+e^{-9(\theta-1/2)})^2} & \text{for } \theta \in [\theta_1, \theta_1'] \cup [\theta_2', \theta_2] \\ 0 & \text{else.} \end{cases}$$

Types in $[\theta_1, \theta_1'] \cup [\theta_2', \theta_2]$ have zero profits. Rents of other types are determined through $\pi_\theta(\theta) = -x(\theta)c_\theta(q(\theta), \theta)$ and $x(\theta)$ is given through the virtual valuation; see lemma 2 in the main text. Using an envelope argument, $VV_\theta(\theta) = -c_\theta(q(\theta), \theta) + \lambda_\theta)c_\theta(q(\theta), \theta) + \lambda(\theta)c_{\theta\theta}(q(\theta), \theta)$. Note that the virtual valuation has a maximum at $\theta' = 0.5$: As $\lambda(0.5) = 0$ and $c_\theta(q(0.5), 0.5) = 0$, it is immediately clear that $VV_\theta(0.5) = 0$. More specifically, for types around 0.5 we get $VV_\theta(\theta) = 2\left(2\theta - 0.5 - \frac{1}{1+e^{-9(\theta-1/2)}}\right) + (0.5-\theta)\frac{9e^{-9(\theta-1/2)}}{\left(1+e^{-9(\theta-1/2)}\right)^2}$ which is positive for θ slightly below 0.5 and negative for θ slightly above 0.5.

Note that $\eta(\theta) \geq 0$ for all types as necessary for optimality. Because of the way we constructed the solution candidate, the sufficient condition for optimality of theorem 1 in Seierstad and Sydsaeter (1987, ch. 5.2) is satisfied. We will argue below that global incentive compatibility is also satisfied and therefore the solution is indeed the optimal contract.

We still have to return to the case $\alpha \leq 8$. As we pointed out above, it is impossible in this case to have types with positive profits within (θ_1, θ_2). The solution will be very similar to the WNM solution in the paper. Types in (θ_1, θ_2) will have zero profits and $q(\theta) = k(\theta)$. Types above θ_2 will have $q(\theta) = q^h(\theta)$ and types below θ_1 will have $q(\theta) = q^l(\theta)$. It is straightforward to check that this solution indeed satisfies all optimality conditions.

[3] λ for types in the zero profit intervals $(\theta_1, \theta_1'] \cup (\theta_2', \theta_2]$ is chosen such that (S14) is satisfied with $q(\theta) = k(\theta)$.

[4] η for types in the zero profit interval is defined by $\eta(\theta) = f(\theta) + \lambda_\theta(\theta)$.

Monotonicity result Monotonicity of the decision variables is of independent interest but also plays a role in second order incentive compatibility. The following lemma provides a useful result.

Lemma S2. *If the relaxed solution satisfies the two properties*

1. $q(\theta) > k(\theta)$ *implies* $q(\theta) \leq q^{fb}(\theta)$

2. $q(\theta) < k(\theta)$ *implies* $q(\theta) \geq q^{fb}(\theta)$

then $q_\theta(\theta) \geq 0$ *and* $X_\theta(\theta)c_\theta(q(\theta), \theta) \leq 0$ *for each* $\theta \in [\underline{\theta}, \bar{\theta}]$.

Proof. To show the second claim of the lemma, we will show that $X_\theta(\theta) \geq 0$ if $q(\theta) > k(\theta)$ and $X_\theta(\theta) \leq 0$ if $q(\theta) < k(\theta)$. If $q(\theta) = k(\theta)$, the claim is trivial as $c_\theta(k(\theta), \theta) = 0$. By the first order condition of q and the definition of the virtual valuation we have

$$\begin{aligned} \text{sign}\{q_\theta(\theta)\} &= \text{sign}\left\{-c_{q\theta}\left(1 - \frac{d\,\lambda(\theta)/f(\theta)}{d\theta}\right) + \frac{\lambda(\theta)}{f(\theta)} c_{q\theta\theta}(q(\theta), \theta)\right\} \\ VV_\theta(\theta) &= -c_\theta(q(\theta), \theta)\left(1 - \frac{d\,\lambda(\theta)/f(\theta)}{d\theta}\right) + \frac{\lambda(\theta)}{f(\theta)} c_{\theta\theta}(q(\theta), \theta). \end{aligned}$$

Note first that the lemma holds for types in a zero profit interval where $q(\theta) = k(\theta)$. As k is increasing, the result on q_θ follows.

Now take a type θ' with positive profits. First assume $\pi_\theta(\theta') \geq 0$. Let θ'' be smallest local π-maximizing type with $\theta'' \geq \theta'$. If $\theta'' = \bar{\theta}$, then $\lambda(\theta'') = 0$ by the transversality condition. If $\theta'' < \bar{\theta}$, we show $\lambda(\theta'') = 0$ in the following way. From $\pi_\theta(\theta'') = 0$, we know that $q(\theta'') = k(\theta'')$ and as θ'' maximizes π we have $q_\theta(\theta'') < k_\theta(\theta'')$. Therefore, λ has to be positive for types slightly below θ'' and negative for types slightly above θ''. We conclude that $\lambda(\theta'') = 0$. From the first order optimality condition for π and the fact that $\pi(\theta) > 0$ for $\theta \in (\theta', \theta'')$, we conclude that $\lambda(\theta') = F(\theta'') - F(\theta')$. MHR implies that the derivative of $(F(\theta'') - F(\theta))/f(\theta)$ is less than 1. The previous steps implied $q(\theta') \geq k(\theta')$ and $\left.\frac{d\,\lambda(\theta)/f(\theta)}{d\theta}\right|_{\theta=\theta'} < 1$. This proves the claim in the lemma for all types at which rents are increasing.

Next, take a type θ' where $\pi_\theta(\theta') < 0$. Let θ'' be the highest type that locally maximizes π such that $\theta'' < \theta'$. Using the same steps as in the previous paragraph, it is possible to show $q(\theta') < k(\theta')$, $\lambda(\theta') = F(\theta'') - F(\theta') < 0$ and $\left.\frac{d\,\lambda(\theta)/f(\theta)}{d\theta}\right|_{\theta=\theta'} < 1$. This proves the claim for types at which rents are decreasing. □

Note that the relaxed solution in the paper (i.e. when assumption 3 is satisfied) satisfies the properties of the previous lemma. Also the relaxed solution in the example above satisfies these properties.

Global incentive compatibility The proofs of lemma 4 and 5 in the appendix of the paper are not using assumption 3 explicitly. However, they use the monotonicity properties above, e.g. $X_\theta(\theta)c_\theta(q(\theta),\theta) \leq 0$ or $q_\theta(\theta) \geq 0$. Therefore, the lemma above gives an easy to check sufficient condition for global incentive compatibility. If the two properties in the lemma are satisfied by the relaxed solution, it will be globally incentive compatible as the proof of lemma 5 in the paper can be readily extended. Hence, the solution in the example above is globally incentive compatible.

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